8

Index of Refraction

The Snell-Descartes law of refraction introduces the index of refraction, n, very early in the development of Optics where, as a matter of fact, it is considered as a normalized value of the speed of propagation, V, of light in a transparent material. The normalization being made with respect to the speed of propagation, c, of electromagnetic waves in a vacuum.

In this chapter, we analyze the physical mechanisms, which indicate that V is different from c, as well as the law of variation of the index of refraction versus the frequency of the electromagnetic waves. First of all, we would like to introduce some important notions that a physicist should bear in mind when he starts to study the index of refraction:

- *Principle of Relativity*: No information can be transmitted faster than the speed of light in a vacuum; more specifically, the electric and magnetic fields that are produced at time t, by a moving electrical charge, cannot be perceived earlier than $(t + r/c)$ at a distance r from the charge.
- Vacuum is the only medium to show *no dispersion*: radio waves, as well as light waves or γ-rays, propagate exactly at the same speed c.
- The phase velocity of a sine wave in a given material depends on the frequency, the variation being more important in the vicinity of the bands of absorption of the considered material.
- Although it's not generally the case, there is no reason why the phase velocity of a wave should not be larger than c, or why the index of refraction should not be smaller than one. For example, the index of refraction of metals for X-ray waves is smaller than unity. This is not contradictory to the relativity principle, since information is carried at the group velocity, which, because of dispersion, is not equal to the phase velocity.

Chapter 8 has been reviewed by Dr. Lionel Bastard from the Institut National Polytechnique, Grenoble.

8.1. Physical Mechanisms Involved with Propagation in a Transparent Material

As far as vacuum propagation is concerned, we will make no attempt to answer the meaningless question: What is vibration? Neither shall we try to find a support to the Maxwell displacement currents that must be introduced, even in a vacuum. Maxwell's equations will be considered as a fundamental hypothesis, fully verified by many experimental consequences. Up to now, Maxwell's equations have never failed.

We will analyze the difference between the propagation in a vacuum and the propagation in a transparent material; the latter being considered as a set (C) of largely submicroscopic particles, bearing either positive charges (nuclei) or negative charges (electrons). C is very often made of subsets with equal numbers of positive and negative charges (atoms or molecules) and also of subsets in which the number of charges of one kind is only slightly larger than the number of charges of the other kind (positive or negative ions).

When a collection of atoms, such as C, receives an electromagnetic wave, an oscillating electromagnetic field, due to this wave, is superimposed on the field of forces that is responsible for the coherency of the corresponding material. It's up to the reader to imagine all the possible interactions. We will only consider the simplest one, that's to say, the action of the electric field of the wave on the charged particles, which is known as *Electric Dipole Interaction* (see equation (1.1)),

$$\frac{d^2\xi}{dt^2}+\gamma\frac{d\xi}{dt}+\omega_0^2\xi=\frac{e}{m}f(t). \tag{8.1}$$

For the sake of simplification, C is limited by a planar interface and the electromagnetic wave is a sine planar wave with a wave vector orthogonal to the interface. The electric charges are put into a forced sine vibration and, in turn, each of them becomes the source of a spherical electromagnetic wave. The electromagnetic field, at a given point P of C, is the addition (interference) of the following fields:

- The field that would exist if C was replaced by vacuum, which is in fact the incident wave.
- The fields that are generated by all the spherical waves.

Because of the symmetry of the problem, the resulting field should have the same value at any point of a plane orthogonal to the direction of propagation. The children that hit the surface of the swimming pool of Figures 1.8 and 1.9 suggest that the superposition of the numerous waves coming from all the elements of C reconstitute a planar wave.

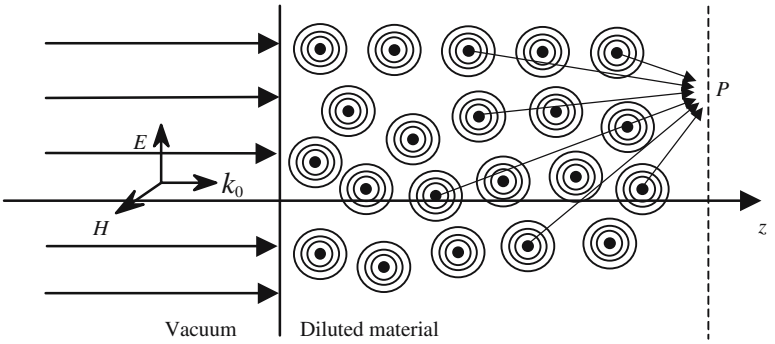


Figure 8.1. Illustration of the propagation of a wave in a material. A planar wave arrives on the planar interface of a piece of material; its electric field sets in motion the electric charges of the atoms, which become oscillating dipoles having the same frequency as the incident wave. The small circles are supposed to suggest the wave surfaces of the spherical wavelets emitted by the dipoles.

8.2. Determination of the Index of Refraction

The determination of the phase velocity of a wave propagating in a transparent material is certainly a difficult problem; at first, the equations of the motion of the charged particles should be integrated and the resulting field at a given point P should be calculated. The next problem would be to find the function written in the right-hand side of equation (8.1). Each of the Q particles of C is submitted to the action of the $Q - 1$ other particles which, at the same time, are submitted to the action of the particle under study. . . . The case of a gas is simplified by the fact that the mutual interactions between the charges are negligible compared to the action of the field of the incident wave.

8.2.1. *Case of a Diluted Material*

We consider a gas made of neutral atoms or molecules. Under the action of the electric field of the incident wave, the electrons vibrate with respect to the nucleus: the gas is then a collection of electrical dipoles, each of them having an electric dipolar momentum that varies sinusoidally versus time. Let N be the number of dipoles per unit volume.

We refer to Figure 8.2, a planar light wave crosses a thin layer of gas, and the light propagates in a vacuum. We want to obtain the expression for the electromagnetic field, $E_{P,\text{total}}$, at a point P located some distance behind the layer of gas. $E_{P,\text{total}}$ can always be considered as the addition of the field $E_{P,0}$ that exists in the absence of the layer of gas and of some extra field $E_{P,\text{slice}}$ that is due to the presence of the layer of gas. Let n and Δz be, respectively, the

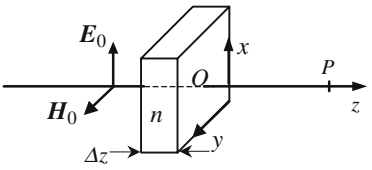


Figure 8.2. Action of a thin film of gas on an electromagnetic wave.

index of refraction and the thickness of the gas sample, we can write the following expressions:

$$E_{P,0} = E_0 e^{j\omega(t-z/c)}, \tag{8.2}$$

$$E_{P,\text{total}} = E_0 e^{j\omega[t-n\Delta z/c-(z-\Delta z)/c]} = E_0 e^{-j\omega(n-1)\Delta z/c} e^{j\omega(t-z/c)}. \tag{8.3}$$

The refractive index of a gas being very close to one, we have

$$e^{-j\omega(n-1)\Delta z/c} \approx 1 - j\omega(n-1)\Delta z/c,$$

$$E_{P,\text{total}} = E_0 e^{j\omega(t-z/c)}[1 - j(n-1)\omega\Delta z/c].$$

The field generated by the layer of gas is thus equal to

$$E_{P,\text{slice}} = -jE_0(n-1)\frac{\omega\Delta z}{c} e^{j\omega(t-z/c)}. \tag{8.4}$$

The gas is considered as a set of oscillating dipoles; let $\eta_0 = N\Delta z$ be the number of dipoles per unit of surface. The field that is created at point P is calculated in Annex 8.A and is given by formula (8.A.11):

$$\boldsymbol{E} = -\boldsymbol{x}j\omega\frac{N\Delta z q x_0}{2\varepsilon_0 c} e^{j\omega(t-z/c)}. \tag{8.5}$$

A comparison of the two formulas (8.4) and (8.5) gives an expression of the refractive index if we are able to relate E_0 and x_0. Because of the thinness of the slice, all the charges vibrate in phase and the function $f(t)$ in equation (8.1) is taken as $f(t) = E_0 e^{j\omega t}$; the solution is then given by

$$\xi = \frac{q}{m}\frac{1}{(\omega_0^2 - \omega^2 + j\gamma\omega)} E_0 e^{j\omega t} \quad \rightarrow \quad x_0 = \frac{q}{m}\frac{E_0}{(\omega_0^2 - \omega^2 + j\gamma\omega)}. \tag{8.6}$$

At last, we obtain the refractive index

$$n = 1 + \frac{Nq^2}{2\varepsilon_0 m(\omega_0^2 - \omega^2 + j\gamma\omega)}. \tag{8.7}$$

A physical interpretation of the index of refraction can be imagined as follows: the electric field of the incident wave polarizes the atoms and sets the electrons in a forced sine motion. The acceleration of the charges radiates a new field that is superimposed on the initial one. The two fields vibrate at the same frequency, however they are not in phase: when interfering they create, inside the material, a field that is not in phase with the incident field.

As the phase difference is proportional to the product of the propagation speed by the length covered inside the material, it seems that the light propagates with a phase velocity that is not equal to c.

8.2.2. *Calculation of the Index of Refraction in a More General Case*

To be able to integrate the equation, we made the assumption of a suitably diluted material so that the oscillations of the electrons were only driven by the incident wave; the mutual interaction of the dipoles was neglected. In a dense material this assumption is no longer valid, so we will come back to Maxwell's equations and make the same kind of development that was made already in Section 2.3. At that time we made no real difference between the propagation in a vacuum or in a transparent dielectric material, except that ε_0 was just replaced by ε.

8.2.2.1. *Polarizability of a Material*

For a more accurate description of the material inside which the propagation occurs, we come back to the definition of the Maxwell displacement vector $\boldsymbol{D}$ and we introduce the polarization vector $\boldsymbol{P}$ that is the dipolar momentum of an unitary volume of the sample. Polarization is of course originated at the atomic level; when one of its electrons is taken away from the equilibrium position by some distance ξ, an atom becomes an electric dipole with an electric dipole momentum equal to $\boldsymbol{p} = q\boldsymbol{\xi}$ (q is the electric charge of the electron). If N is the cubic density of atoms, we have

$$\boldsymbol{P} = N\boldsymbol{p} = Nq\boldsymbol{\xi}. \tag{8.8}$$

In the frame of a linear interaction between the material and the light, $\boldsymbol{\xi}$ is proportional to the electric field $\boldsymbol{E}$, the proportionality coefficient is called the *atomic polarizability* ϖ and is defined as

$$\boldsymbol{p} = \varpi\varepsilon_0\boldsymbol{E}. \tag{8.9}$$

Let l be the number of electrons that, inside a given atom, are concerned with the interaction with a light wave. A simple model of a harmonic oscillator is chosen, with an eigenfrequency $\omega_{0,k}$ and a damping coefficient γ_k, k is a labeling index ($0 < k < l$),

$$m(\ddot{\xi}_k + \gamma_k\dot{\xi}_k + \omega_{0,k}^2\xi_k) = qE_0e^{j\omega t} \quad \rightarrow \quad \xi_k = \frac{q}{m}\frac{1}{\Delta_k(\omega)}E_0e^{j\omega t},$$

with

$$\Delta_k(\omega) = -\omega^2 + j\gamma_k\omega + \omega_{0,k}^2. \tag{8.10}$$

The variation of the atomic polarizability versus frequency is described by

$$\varpi(\omega) = \frac{q^2}{\varepsilon_0 m} \sum_k \frac{1}{\Delta_k} = \frac{q^2}{\varepsilon_0 m} \sum_k \frac{1}{-\omega^2 + j\gamma_k \omega + \omega_{0,k}^2}. \tag{8.11}$$

Oscillator Strength

We should be reasonably worried by the previous method of establishing the equations, which is at the same time pragmatic and relaxed, since we used a very classical treatment for microscopic phenomena that fall, in principle, into the scope of Quantum Mechanics. A full quantum treatment is possible and gives very similar results, except for the introduction of a set of real coefficients, called the *oscillator strengths* f_k, which are characteristic of the atoms under consideration,

$$\varpi(\omega) = \frac{q^2}{\varepsilon_0 m} \sum_k \frac{f_k}{\Delta_k} = \frac{q^2}{\varepsilon_0 m} \sum_k \frac{f_k}{-\omega^2 + j\gamma_k \omega + \omega_{0,k}^2}, \tag{8.12.a}$$

with

$$0 < f_k < 1 \quad \text{and} \quad \sum_k f_k = 1. \tag{8.12.b}$$

Electrical Susceptibility

When the polarization results from the application of an electric field $\boldsymbol{E}$, the electrical susceptibility χ is introduced by the following formula:

$$\boldsymbol{P} = \varepsilon_0 \chi \boldsymbol{E}. \tag{8.13}$$

Going back to formulas (8.11) and (8.13) and introducing the cubic density of electrons N_k, we obtain

$$\boldsymbol{P} = \varepsilon_0 \sum_k N_k \varpi(\omega) \boldsymbol{E} = \varepsilon_0 \chi(\omega) \boldsymbol{E},$$

$$\chi(\omega) = \sum_k N_k \varpi(\omega) = \frac{q^2}{\varepsilon_0 m} \sum_k N_k \frac{f_k}{\Delta_k(\omega)}. \tag{8.14}$$

The variation of the polynomial $\Delta_k = -\omega^2 + j\gamma_k\omega + \omega_{0,k}^2$ is rather smooth, except in the close vicinity of a band of absorption, when $\omega \approx \omega_{0,k}$. The damping coefficient γ_k is much smaller than the eigenfrequency, so the roots of the polynomial are almost equal to $\omega_{0,k}$ and $\Delta_k(\omega)$ can be simplified as

$$\begin{aligned} \Delta_k(\omega) &= \omega_{0,k}^2 - \omega^2 + j\gamma_k\omega = (\omega_{0,k} + \omega)(\omega_{0,k} - \omega) + j\gamma_k\omega \\ &\approx 2\omega_{0,k}\left[(\omega_{0,k} - \omega) + j\frac{\gamma_k}{2}\right]. \end{aligned}$$

Formula (8.14) becomes

$$\chi(\omega) = \frac{q^2}{2\varepsilon_0 m} \sum_k \frac{N_k f_k}{\omega_{0,k}} \frac{1}{(\omega_{0,k} - \omega + j\gamma_k/2)},$$

setting $\chi(\omega) = \chi'(\omega) - j\chi''(\omega)$,

$$\chi'(\omega) = \frac{q^2}{2\varepsilon_0 m} \sum_k \frac{N_k f_k}{\omega_{0,k}} \frac{(\omega_{0,k} - \omega)}{\left[(\omega_{0,k} - \omega)^2 + \gamma_k^2/4\right]}, \tag{8.15.a}$$

$$\chi''(\omega) = \frac{q^2}{2\varepsilon_0 m} \sum_k \frac{N_k f_k}{\omega_{0,k}} \frac{\gamma_k/2}{\left[(\omega_{0,k} - \omega)^2 + \gamma_k^2/4\right]}. \tag{8.15.b}$$

Law of Refraction of the Index of Refraction

The square of the index of refraction is equal to the square of the dielectric constant ε_0. In a dielectric material, the simple relationship $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E}$, that is valid in a vacuum, becomes $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$. When the existence of the polarization is the result of the action of an external electric field, we can write the following equations:

$$\boldsymbol{P} = \varepsilon_0 \chi \boldsymbol{E} \quad \rightarrow \quad \boldsymbol{D} = \varepsilon_0 (1 + \chi)\boldsymbol{E} = \varepsilon \boldsymbol{E} \quad \rightarrow \quad \varepsilon = \varepsilon_0 (1 + \chi), \tag{8.16.a}$$

$$\varepsilon_r = \frac{\varepsilon}{\varepsilon_0} = n^2, \quad n^2 = 1 + \chi(\omega) = 1 + N\varpi(\omega), \tag{8.16.b}$$

$$n^2 = 1 + \frac{Nq^2}{\varepsilon_0 m} \sum_k \frac{f_k}{\Delta_k(\omega)}. \tag{8.17.a}$$

In the case of a diluted material the refractive index is of the order of one, formula (8.17.a) then takes a simpler form that coincides with (8.7),

$$n \approx 1 + \frac{\chi}{2} \quad \rightarrow \quad n' \approx 1 + \frac{\chi'}{2} \quad \text{and} \quad n'' \approx 1 + \frac{\chi''}{2}. \tag{8.17.b}$$

8.2.2.2. *Local Field Correction (Lorentz-Lorenz or Clausius-Mossotti Formula)*

The method for establishing formula (8.15) is no longer limited by the necessity that the index of refraction is close to unity; however, it contains the implicit assumption that the incident field is the *polarizing field*, that's to say, the field that creates the dipolar oscillation. Reasonable in the case of a diluted material, this hypothesis is not true in a dense material where the field that exists in the environment of an atom, also called a *local field*, is the superposition of the applied field and of a *depolarizing field*, which corresponds to the action of all the other atoms.

The same problem is also met in Electrostatics, when it is desired to calculate the electric field inside a dielectric material. A good description may be found in a standard textbook, see, for example, *Solid State Physics* by

Charles Kittel. To evaluate the depolarizing field, we imagine that the atom is inside a small cavity dug inside the material.

In the case of an isotropic or a cubic material, the cavity can be a sphere. To obtain the local field $\boldsymbol{E}_{\text{local}}$ inside the cavity, a term equal to $\boldsymbol{P}/3\varepsilon_0$ should be added to the external field $\boldsymbol{E}$,

$$\boldsymbol{E}_{\text{local}} = \boldsymbol{E} + \frac{\boldsymbol{P}}{3\varepsilon_0}. \tag{8.18}$$

We will do the calculation in the simpler case where there is only one electron per atom that contributes to the index of refraction,

$$\boldsymbol{P} = \varepsilon_0 N\varpi \boldsymbol{E}_{\text{local}},$$

$$\boldsymbol{P} = \varepsilon_0 \chi \boldsymbol{E} \quad \rightarrow \quad \chi = \frac{3N\varpi}{3 - N\varpi},$$

$$\varepsilon_r = \frac{\varepsilon}{\varepsilon_0} = n^2 = 1 + \chi \quad \rightarrow \quad \frac{n^2 - 1}{3} = \frac{N\varpi}{3 - N\varpi},$$

$$\frac{n^2 - 1}{n^2 + 2} = \frac{1}{3} N\varpi. \tag{8.19}$$

When more than one electron per atom contributes to the index of refraction, formula (8.19) is generalized as

$$\frac{n^2 - 1}{n^2 + 2} = \frac{1}{3} \sum_k N_k \varpi_k. \tag{8.20}$$

Formula (8.20) is called the *Lorentz-Lorenz* or also *Clausius-Mossoti formula*. Its demonstration explicitly admits a cubic environment of the atoms.

Gladstone's Law

For a gas, formula (8.20) becomes simpler and coincides with (8.7). If the gas is made of only one kind of molecule, the right-hand term is proportional to the number of molecules per unit volume, that's to say, to the density d. In this case the Lorentz-Lorenz formula is written as

$$\frac{(n-1)}{d} = \text{constant}. \tag{8.20′}$$

Known as the *Gladstone law*, (8.20′) shows that the index of refraction increases with pressure, which makes the material denser, and decreases with temperature. Previous remarks remain true in the case of a liquid or solid-state material. The index variation that can be produced by a reasonable variation of the temperature remains weak (up to three decimal places); the temperature effect is however a convenient way to finely tune the value of the refractive index.

The acoustooptic effect is an interesting consequence of the effect of pressure. The propagation of an acoustic wave inside (sonic, ultrasonic, or hypersonic) a material modulates the index of refraction at the same frequency. If a light beam and an acoustic wave, with wavelengths of the same order of magnitude, propagate simultaneously in a liquid or a solid, the light can be efficiently diffracted by the index grating formed by the acoustic wave.

8.3. The Index of Refraction Is a Complex Number

According to formulas (8.16) and (8.17) the dielectric constant and the index of refraction are complex numbers:

$$\begin{aligned} &n = n' - jn'', \quad \varepsilon = \varepsilon_0 \varepsilon_r = \varepsilon' - j\varepsilon'' = \varepsilon_0(\varepsilon_r' - j\varepsilon_r''), \\ &\varepsilon_r = n^2 \quad \rightarrow \quad \varepsilon_r' = n'^2 - n''^2 \quad \text{and} \quad \varepsilon_r'' = 2n'n''. \end{aligned} \tag{8.21}$$

Negative signs have been introduced in the above formulas, because in most cases (sample at thermal equilibrium, for example) the imaginary parts of the index and of the dielectric constant are negative.

8.3.1. *Beer's Law*

The imaginary part of the refractive index corresponds to *attenuation*, or possibly *amplification*, of the wave during propagation. It is convenient to define a complex wave vector:

$$\begin{aligned} &\boldsymbol{k} = \boldsymbol{k}' - j\boldsymbol{k}'' \quad \text{with} \quad |\boldsymbol{k}'| = n'\omega/c \quad \text{and} \quad |\boldsymbol{k}''| = n''\omega/c, \\ &\boldsymbol{E} = \boldsymbol{E}_0 e^{j\omega(t-z/c)} = \boldsymbol{E}_0 e^{-n''\omega z/c} e^{j\omega(t-n'z/c)} = \boldsymbol{E}_0 e^{-k''z} e^{j(\omega t - k'z)}, \end{aligned} \tag{8.22}$$

where $e^{j\omega(t-n'z/c)}$ is a complex number that keeps a constant modulus during the propagation. On the contrary $e^{-\omega n''z/c}$ is a real number that decreases if n'' is positive. The expression of the intensity is given by

$$I = I_0 e^{-2k''z} = I_0 e^{-\alpha z} \quad \text{with} \quad \alpha = 2k''. \tag{8.23}$$

The relationship (8.23) is called Beer's law of variation of the intensity during the propagation of a light beam in an absorbing material. The coefficient α is called the absorption coefficient; its variation $\alpha(\omega)$ versus frequency is the absorption spectrum of the material and is easily obtained from spectroscopic measurements.

Beer's law can also be written using the logarithm of the intensity

$$I_{\text{db}} = 10\log_{10}\left(\frac{I}{I_0}\right) = -(10\alpha\log_{10})z = -\alpha_{\text{db}} z,$$

$\alpha_{\text{db}} = 10\alpha\log_{10}(e) = 4.3\alpha$ is the "db-coefficient of absorption."

8.3.1.1. *Order of Magnitude*

- For a standard piece of glass, in the middle of the visible spectrum ($\lambda = 0.6\,\mu$m), the real part of the index is 1.5 and the absorption coefficient is of the order of some db/cm,

$$\alpha_{\text{db}} \approx 0.1\,\text{db/cm} = 10\,\text{db/m} \quad \rightarrow \quad \alpha = 10/4.3 = 2.3\,\text{m}^{-1} \quad \rightarrow \quad k'' = \alpha/2 = 1.15\,\text{m}^{-1},$$
$$n'' = k''\lambda/2\pi = 0.11.10^{-6} \quad \rightarrow \quad n = 1.5 - j0.11.10^{-6}.$$

- For silica, which is probably one of the most transparent materials in the near infrared ($1.55\,\mu$m), $\alpha_{\text{db}} \approx 0.15$ db/km = $0.15.10^{-3}$ db/m.
- For a rather absorbing material $\alpha_{\text{db}} \approx$ some db/mm. An absorption of one decibel per wavelength is very high.

Remark

The imaginary part of the index comes from the phenomenological damping coefficient in the equation of the harmonic oscillator that describes the motion of an electron. As with any damping coefficient it corresponds to some energy dissipation: it's the reason why the wave is attenuated. Our model says nothing of what happens with the dissipated energy, the answer to this difficult question would need a more accurate description of the phenomena. The lost energy will often appear as heat, corresponding to erratic vibrations of the atoms, but it can also appear as stray light diffused around the propagation medium (see, for example, "The Reason Why the Sky Is Blue" in the Annex of Chapter 11).

8.3.2. *Index of Refraction of a Metal*

The phenomenological model of equation (8.1) corresponds to a dielectric material inside which the electrons are bound by a restoring force associated to the term $\omega_{\text{atom.}}^2 \xi$. The case of a metal is different: beside bound electrons, as in the case of a dielectric, *free electrons* can move all over the sample. The phenomenological equation of the motion of a free electron is almost identical to (8.1), except that the restoring force is withdrawn,

$$m\left(\ddot{\xi} + \gamma_k \dot{\xi}\right) = qE_0 e^{j\omega t}.$$

For free electrons formula (8.17.a) is to be replaced by

$$n^2 = 1 + \frac{Nq^2}{\varepsilon_0 m} \frac{1}{\left(-\omega^2 + j\gamma\omega\right)}. \tag{8.24}$$

The index of refraction of a metal has in principle two components, however the contribution of the bound electrons is negligible as compared to the contribution of free electrons. Free electrons, by definition, can be found

anywhere in the metal and they see an *average electric field*, and there is no necessity to make a local field correction.

Plasma Frequency

The dissipation of energy is due to collisions of the free electrons with the atoms. An important parameter in this case is the mean free path that is the average distance covered by an electron between two collisions; the collision time τ, which is the time necessary to cover this distance, is typically 10^{-13} s, about ten times the optical period ($\omega\tau \cong 10$). The conductivity σ of the metal is linked to the collision time and it can be established that the square of the index is given by

$$n^2 = 1 + \frac{\sigma/\varepsilon_0}{j\omega(1 + j\omega\tau)}. \tag{8.24'}$$

If $\omega\tau$ is considered to be much greater than one, (8.24′) becomes

$$n^2 = 1 - \frac{\sigma}{\varepsilon_0\omega^2\tau} = 1 - \frac{\omega_p^2}{\omega^2} = 1 - \frac{\lambda^2}{\lambda_p^2}, \tag{8.25}$$

where ω_p and λ_p are, respectively, the plasma frequency and plasma wavelength of the metal; for most metals they belong to the ultraviolet or X-band domains. The optical frequencies are much smaller than the plasma frequencies, and so the square of the index of refraction is negative. As a first approximation the index of a metal is purely imaginary, which means that metals are always very absorbing for optical waves.

The modulus of the refractive index of metals has always had a high value; as a consequence, the reflection coefficient is also very high, which explains the typical sparkling appearance of nonoxidized metals. In reality, the indices of metals are complex numbers, with an imaginary part larger than the real part. At a frequency higher than the plasma frequency, in the ultraviolet or X-band, the index is real and smaller than unity.

8.3.3. *The Kramers-Krönig Formula*

When a light beam is incident on an interface separating a vacuum from some material, a reflected beam and a transmitted beam are generated. The

Table 8.1. Complex indices of some metals at $\lambda = 589.3\,\text{nm}$.

Sodium	$0.044 - j2.42$	Tin	$1.48 - j5.25$
Silver	$0.2 - j3.44$	Gold	$0.47 - j2.83$
Aluminum	$1.44 - j5.23$	Copper	$0.62 - j2.57$

material can be considered as a *system* that is submitted to an *excitation* by the incident beam and then gives a *response* that is made of the reflected and transmitted beams. As these two beams are a consequence of the forced vibrations of the dipoles of the material, the polarization of the material can also be considered as the answer to the system of the excitation of the incoming light. The existence and main properties on the optical index come from the fact that the excitation and response are linked by linear equations; these equations are, at first, the relation (8.13) between $\boldsymbol{P}$ and $\boldsymbol{E}$, and second, the differential equation (8.1) that governs the motion of electrons.

Because of the linearity, a sine excitation produces a proportional sine response; the proportionality coefficient is a complex number that varies with the frequency. This is the reason why the permittivity, as well as the index of refraction, are dispersive complex quantities. We are now going to show that an important consequence of the principle of causality is the existence of two necessary relations between the frequency variation laws of the real and imaginary parts of the index of refraction. These relations were introduced independently by Krönig in 1926 and by Kramers in 1927; known as the Kramers-Krönig formulas, they establish relations between the dispersion laws of the real part, $\chi'(\omega)$, and the imaginary part, $\chi''(\omega)$, of the susceptibility.

The physical reason for the Kramers-Krönig formulas comes from the fact that, because of the causality principle, the response cannot happen before the excitation. In our case, the electrons cannot start to oscillate prior to the arrival of the incident beam. This mathematical demonstration is rejected in Annex 8.B; the result is given by formula (8.26):

$$\chi'(\omega)=\frac{2}{\pi}\int_0^{+\infty}\frac{\omega'\chi''(\omega')}{(\omega'^2-\omega^2)}d\omega' \quad \text{and} \quad \chi''(\omega)=-\frac{2\omega}{\pi}\int_0^{+\infty}\frac{\chi'(\omega')}{(\omega'^2-\omega^2)}d\omega'. \qquad (8.26)$$

The examination of (8.26) immediately shows a mathematical difficulty since the denominators of the fractions to be integrated cancel if $\omega=\omega'$. A mathematical trick should be used considering the principal part of the integrals (see Annex 8.B). The interesting point is that, once the variation law $\chi'(\omega)$ has been measured, or calculated, $\chi''(\omega)$ can be obtained from (8.26), and vice versa. However, one must be aware that (8.26) is a *relationship between two functions* which has two consequences:

(i) the knowledge of the value of $\chi'(\omega_0)$ at a given frequency ω_0 doesn't allow any prediction about $\chi''(\omega_0)$;
(ii) to obtain $\chi''(\omega)$ from (8.26) requires, in principle, knowing the variations of $\chi'(\omega)$ over the whole frequency range, from zero to infinity.

8.3.4. *Analytical Expression of the Laws of Dispersion*

The index of refraction can be measured with high accuracy over all of the visible spectrum; as this knowledge is very important in many cases, various empirical formulas have been established to describe its variation versus

frequency. These formulas are quite accurate and give the value of the refractive index up to five decimal places.

Cauchy's formula: This is valid for materials that have only ultraviolet absorption bands,

$$n = A + \frac{B}{\lambda^2} + \frac{C}{\lambda^4}.$$

Briot's formula: This is valid for materials with infrared and ultraviolet absorption bands,

$$n = -A'\lambda^2 + A + \frac{B}{\lambda^2} + \frac{C}{\lambda^4}.$$

The previous formulas are nothing other than series developments of the Lorentz-Lorenz formula. They concern materials that are transparent in the visible and are valid far from the absorption bands.

Normal dispersion: Inside a transparency band, the index of refraction is a decreasing function of the wavelength or, which is the same, an increasing function of the frequency. In Annex 8.B, this proposition will be established as a consequence of the Kramers-Krönig formulas. It's convenient to remember that the refractive index is higher in the blue than in the red.

$n_{\text{blue}} > n_{\text{red}}$: a blue ray is more deviated by a prism than a red one.

A frequency domain inside of which the index increases with frequency is sometimes called a *zone of normal dispersion.*

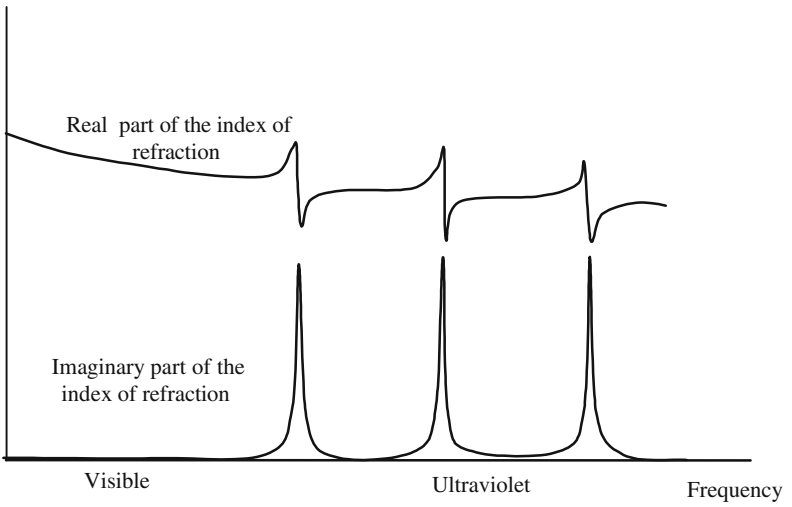


Figure 8.3. Dispersion and absorption curves of a material that is transparent in the visible and has three absorption bands in the ultraviolet.

Anomalous dispersion: When the frequency falls inside an absorption band, the index variation is no longer monotonic; one then speaks of *a zone of anomalous dispersion.*

8.4. Index of Refraction and Populations of the Energy Levels of a Transition

This section should be read in parallel with Section 9.2.3 where the absorption is related to the transitions made by an atom between the two energy levels associated to a dipolar transition (cf. formula (9.31)); the absorption coefficient α is related to the transition parameters by the following formula:

$$\alpha = \frac{c^2}{8\pi\nu^2 u} A f(\nu)(N_0 - N_1) = \frac{c^2}{8\pi\nu^2 u} \frac{1}{\tau_{\text{radiative}}} f(\nu)(N_0 - N_1). \qquad (8.27)$$

The expression of the imaginary part of the susceptibility, $\chi''(\nu)$, can be deduced from the absorption coefficient using the following expressions that come from equations (8.17.b) and (8.23):

$$\begin{aligned} n'' &= \frac{\chi''}{2}; \quad \alpha = 2k'' = \chi'' \frac{\omega}{c}, \\ \chi''(\nu) &= \frac{c^3}{16\pi^2\nu^3} n'^2 \frac{1}{\tau_{\text{radiative}}} f(\nu)(N_0 - N_1). \end{aligned} \qquad (8.28)$$

It is left as an exercise to use the Kramers-Krönig formulas to deduce the expression of the real part, $\chi'(\nu)$, of the susceptibility from (8.28) in the case

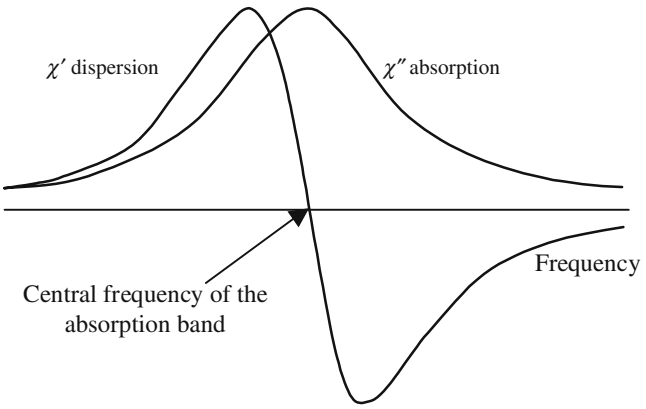


Figure 8.4. Representation of the variations χ' and χ'' versus frequency, in the case of a Lorentzian profile. For the central frequency, the absorption is maximum while χ' changes its sign.

of a Lorentzian expression of the function $f(\nu)$. It's easier to make the ratio of the two formulas (8.15.a) and (8.15.b):

$$\chi'(\nu) = \frac{2(\omega_{\text{atom}} - \omega)}{\lambda}\chi''(\nu),$$
$$\chi'(\nu) = \frac{c^3}{8\pi^2\nu^3} n'^2 \frac{(\omega_{\text{atom}} - \omega)}{\gamma} f(\nu)(N_0 - N_1). \tag{8.29}$$

Annex 8.A

Electric Dipole Radiation

Any piece of material is in fact a collection of electric dipoles that can freely oscillate or that are forced to oscillate by an incoming electromagnetic wave. The purpose of this annex is to calculate the electromagnetic field radiated from an oscillating dipole.

8.A.1. Definition of an Oscillating Dipole

We refer to Figure 8.A.1 and consider a dipole made of two electric charges, $+q$ and $-q$, disposed at points Q_+ and Q_- separated by a distance d; by definition, the electric dipolar momentum $\boldsymbol{p}$ of the dipole is the product of the absolute value of the charges by the vector $\boldsymbol{a} = \boldsymbol{Q}_-\boldsymbol{Q}_+$,

$$\boldsymbol{p} = q\boldsymbol{Q}_+\boldsymbol{Q}_-.$$

If $\boldsymbol{p}$ is a constant vector, the dipole is said to be static; an oscillating dipole corresponds to the case where the distance follows a sine time variation:

$$\boldsymbol{p} = \boldsymbol{p_0}\cos\omega t = \mathrm{Re}[\boldsymbol{p}_0 e^{j\omega t}]. \tag{8.A.1}$$

8.A.2. Electric and Magnetic Fields Created by a Dipole

We will first successively consider the case of a static dipole and then of an oscillating dipole.

8.A.2.1. *Static Dipole*

The determination of the electric field created by a static dipole is a trivial electrostatic calculation: the field at point P in Figure 8.A.1 is parallel to the plane that contains point P and vector $\boldsymbol{p}_0$; using the unitary vectors $\boldsymbol{r}_1$ and $\boldsymbol{\theta}_1$, the expression of the field is

$$\boldsymbol{E} = \frac{p_0}{4\pi\varepsilon_0}\frac{1}{r^3}(2\boldsymbol{r}_1\cos\theta + \boldsymbol{\theta}_1\sin\theta_1). \tag{8.A.2}$$

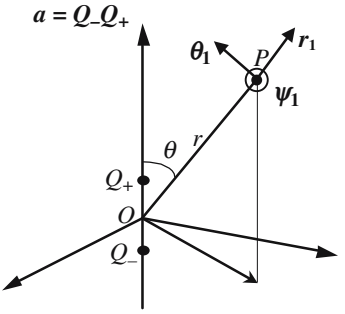


Figure 8.A.1. Oscillating dipole. ($\boldsymbol{r}_1$, $\boldsymbol{\theta}_1$, $\boldsymbol{\psi}_1$) is a unitary direct tetrahedron; $\boldsymbol{r}_1$ is parallel to $\boldsymbol{a}$, $\boldsymbol{\theta}_1$ is in the plane (OP, a), $\boldsymbol{\psi}_1$ is orthogonal to the two other vectors.

8.A.2.2. *Oscillating Dipole*

An oscillating dipole is, at the same time, a pair of charges and an electric current, since the charges are moving; and so an electric field and a magnetic field are generated. The calculation of this electromagnetic field is interesting, from a physical point of view; the Theory of Relativity is implicitly involved, since the effect of a change in the positions of the charges cannot be felt instantaneously at point P. This calculation is also trivial in Electromagnetics courses. The components of the field can be deduced from a scalar potential V and from a vector potential $\boldsymbol{A}$,

$$V = \frac{q}{4\pi\varepsilon_0}\left(\frac{e^{j\omega(t-Q_+P/c)}}{Q_+P} - \frac{e^{j\omega(t-Q_-P/c)}}{Q_-P}\right) \approx \frac{qa}{2\varepsilon_0\lambda}\frac{1}{r}\cos\theta\left(\frac{\lambda}{2\pi r} + j\right)e^{j\omega(t-r/c)},$$

$$\boldsymbol{A} = \frac{\mu_0}{4\pi r}\, j\omega qa e^{j\omega(t-OP/c)}(\boldsymbol{r}_1\cos\theta - \boldsymbol{\theta}_1\sin\theta). \tag{8.A.3}$$

$$\boldsymbol{E} = \frac{\pi qa}{\varepsilon_0\lambda^2 r}e^{j\omega(t-r/c)}\left[2\boldsymbol{r}_1\left(\frac{\lambda^2}{16\pi^2r^2} + j\frac{\lambda}{2\pi r}\right)\cos\theta + \boldsymbol{\theta}_1\left(\frac{\lambda^2}{16\pi^2r^2} + j\frac{\lambda}{2\pi r} - 1\right)\sin\theta\right], \tag{8.A.4.a}$$

$$\boldsymbol{H} = \frac{1}{\mu_0}\nabla\wedge A = -\boldsymbol{\psi}_1\sin\theta\frac{\pi cqa}{r\lambda^2}e^{j\omega(t-r/c)}\left(j\frac{\lambda}{2\pi r} - 1\right). \tag{8.A.4.b}$$

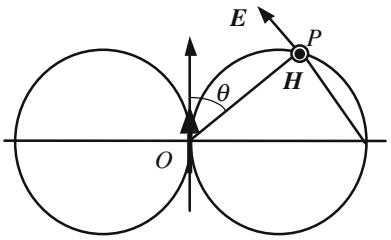
In general, the spatial field distributions are quite complicated, they become simpler if:

- $a \ll \lambda = 2\pi c/\omega$, the distance between the charges is small as compared to the wavelength.
- The point P is far from the dipole.

If the frequency goes to zero (or if the wavelength goes to infinity), formula (8.A.4.a) becomes identical to (8.A.2). At a long distance from the dipole and keeping only the $1/r$ terms, the following formulas are obtained:

$$\boldsymbol{E} = -\frac{\pi qa}{\varepsilon_0\lambda^2}\frac{e^{j\omega(t-r/c)}}{r}\sin\theta\boldsymbol{\theta}_1, \tag{8.A.5}$$

$$\boldsymbol{H} = -\frac{\pi cqa}{\lambda^2}\frac{e^{j\omega(t-r/c)}}{r}\sin\theta\boldsymbol{\psi}_1. \tag{8.A.6}$$



$$p = |\boldsymbol{p}_0|,$$
$$\boldsymbol{E} = \frac{-\pi p_0}{\varepsilon_0 \lambda^2} \frac{1}{r} e^{j\omega(t-r/c)} \sin\theta \boldsymbol{\theta}_1,$$
$$\boldsymbol{H} = \frac{-\pi c p_0}{\lambda^2} \frac{1}{r} e^{j\omega(t-r/c)} \sin\theta \boldsymbol{\psi}_1.$$

Figure 8.A.2. Diagram of emission of a dipole. The field keeps a constant modulus along a circle that is tangent to the dipole at point *O*. No emission in the direction of the dipole.

The expression $e^{j\omega(t-r/c)}/r$ in (8.A.5) and (8.A.6) corresponds to the propagation of a wave. (***E***, ***H***, ***OP***) is a direct trihedron. The ratio of the modulus of the electric and magnetic fields remains constant and is equal to the wave impedance

$$\frac{|\boldsymbol{E}|}{|\boldsymbol{H}|} = \frac{1}{\varepsilon_0 c} = \mu_0 c = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377 \text{ ohms}. \tag{8.A.7}$$

The $1/r$ law, which must be associated to the energy conservation, corresponds to a *slow diminution* of the amplitude of the wave as a function of the distance of propagation; the rate of decrease would be much faster if it were an exponential law. This is the explanation of the enormous range of the waves that are emitted by an oscillating dipole. The radio wave coming from a modest emitter (a few watts) can still be detected after having covered several thousands of kilometers; light rays coming from the very edge of the universe are seen with the naked eye. . . .

Polarization and Angular Repartition of the Radiation of a Dipole

The light radiated by a dipole is linearly polarized; at a given point *P*, the electric field is parallel to the plane, that is defined by *P* and the electric momentum ***p*** of the dipole, and is perpendicular to the line joining *P* to the center of the dipole.

Despite the presence of $e^{j\omega(t-r/c)}/r$ in the expression for the fields, even at a great distance, the radiation of a dipole is quite different from a spherical wave. The indicatrix of emission doesn't have any spherical symmetry, since the amplitude cancels in the direction of the dipole. From a mathematical point of view, the cancellation comes from the presence of $\sin\theta$ in formulas (8.A.5) and (8.A.6); the physical interpretation comes from the fact that an electromagnetic wave must be transversal and that, because of the cylindrical symmetry of the problem, the fields cannot have components that would be orthogonal to the dipole.

8.A.3. Power Radiated by an Oscillating Dipole

We will give two methods for calculating the power that is radiated by an oscillating dipole. The first method is fully electromagnetic and relies on the previous expressions of the fields and on the Poynting theorem. The second will use results of the Theory of Relativity: it's because of the acceleration of the moving charges that energy is radiated by the dipole.

8.A.3.1. *Calculation Using the Poynting Theorem*

The power that is radiated through a given surface Σ is equal to the flux of the Poynting vector $\boldsymbol{\Pi}$ through the surface,

$$\boldsymbol{\Pi} = \tfrac{1}{2}\,\mathrm{Re}[\boldsymbol{E} \wedge \boldsymbol{H}^*].$$

In the above formula, the asterisk is for complex conjugation; its presence is the time to disappear ($e^{j\omega t}e^{-j\omega t}$) in the vector product; from a physical point of view this means that the power is averaged over many periods,

$$P = \iint_{\Sigma} \boldsymbol{\Pi}\, d\boldsymbol{S} = \frac{\pi^2 c p_0^2}{2\varepsilon_0 \lambda^4} \iint_{\Sigma} \frac{\sin^2\theta}{r^2}\, r^2 \sin\theta\, d\theta\, d\varphi.$$

To evaluate the integral, Σ is a sphere with the dipole at the center,

$$P = \frac{\pi^2 c p_0^2}{2\varepsilon_0 \lambda^4} \int_0^{2\pi} d\varphi \int_0^{\pi} \sin^3\theta\, d\theta = \frac{p_0^2 \omega^4}{12\pi\varepsilon_0 c^3} = \text{constant} \quad (\forall\, R). \tag{8.A.8}$$

- P is independent of the radius of the sphere, which ensures the conservation of energy.
- P varies as the fourth power of the frequency or, similarly, as the fourth power of the inverse of the wavelength. For a given dipolar momentum, the radiated power is $2^4 = 16$ times larger in the blue ($0.4\,\mu$m) than in the red ($0.8\,\mu$m).

8.A.3.2. *Calculation Using Relativity Considerations*

Richard Feynman gave an elegant and didactic presentation in the chapter entitled "Electromagnetic Radiation" of his book *Lectures on Physics*.

Formulas (8.A.9) give the expressions of the electric and magnetic fields that are created by a charge q moving along some trajectory Γ,

$$\boldsymbol{E} = \frac{-q}{4\pi\varepsilon_0}\left(\frac{\boldsymbol{e}_{r'}}{r'^2} + \frac{r'}{c}\frac{d}{dt}(\boldsymbol{e}_{r'}) + \frac{1}{c^2}\frac{d^2}{dt^2}(\boldsymbol{e}_{r'})\right), \tag{8.A.9.a}$$

$$\boldsymbol{H} = \frac{-1}{\mu_0 c}\boldsymbol{e}_{r'} \wedge \boldsymbol{E}. \tag{8.A.9.b}$$

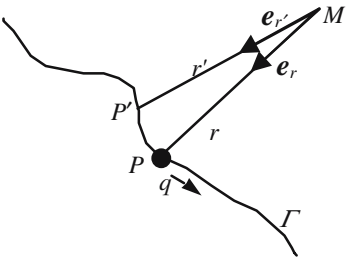


Figure 8.A.3. An electric charge q is moving along the trajectory Γ; $\boldsymbol{e}_r$ and $\boldsymbol{e}_{r'}$ are the respective unit vectors of MP and MP'.

The two first terms of (8.A.9.a) are the relativistic expression of the Coulomb law for the electric field created by a charge q. *The last term is associated to the electromagnetic field that is radiated by the oscillating dipole*; to obtain its expression, we need to know the acceleration of the extremity of the unit vector $\boldsymbol{e}_{r'}$. The electromagnetic field that is radiated, at a given point P and at a given time t, may be expressed as a function of the acceleration of the charge at time $(t - r/c)$.

The following important results are obtained:

A charge with a uniform rectilinear motion doesn't radiate any field.
A charge radiates electromagnetic energy, only if it is accelerated.
The total radiated power is proportional to the square of the acceleration.

Consequences

- In the case of harmonic oscillations, the acceleration is proportional to the square of the angular frequency ω, and thus the radiated power is proportional to ω^4, which agrees with formula (8.A.8).
- *Synchrotron radiation*: A charged particle orbiting along the trajectory of some accelerator (a synchrotron, for example) is permanently submitted to a centripetal acceleration and radiates an electromagnetic field, which is called synchrotron radiation. According to the working conditions of the accelerator, the radiated spectrum covers the whole electromagnetic domain, including visible, ultraviolet, and X-rays.

8.A.3.3. *Lifetime of the Excited Level of an Electric Dipole Transition*

We consider an atom that is isolated in a vacuum; an electron of this atom and the nucleus makes an electric dipole, their distance $\xi(t)$ follows the equation of a harmonic oscillator:

$$\frac{d^2\xi}{dt^2} + 2\alpha\frac{d\xi}{dt} + \omega_{\text{atom}}^2\xi = 0.$$

After having been taken away from its equilibrium position, the electron will do free oscillations, let $W(t)$ and W_0, respectively, be the values of the "mechanical energy" of the oscillator and consider that the damping is weak ($\omega_{\text{atom}} \gg \alpha$):

$$\begin{aligned} \xi(t) &= \xi_0 e^{-\alpha t}\cos\omega_{\text{atom}}t, \\ W(t) &= W_0 e^{-2\alpha t} = W_0 e^{-t/\tau_{\text{radiative}}}, \end{aligned} \qquad \text{with} \quad \tau_{\text{radiative}} = 1/2\alpha.$$

The energy $W(t)$ that is stored by the oscillator decreases as time passes; the instant power that is dissipated is given by

$$\begin{aligned} P(t) &= -\frac{dW}{dt} = 2\alpha W(t), \\ 2\alpha &= \frac{1}{\tau_{\text{radiative}}} = \frac{P(t)}{W(t)}, \end{aligned}$$

where τ_{rad} is called *the radiative lifetime* of the oscillator. To obtain an expression of this time constant, we will admit that the dissipation of energy corresponds to the radiated power that has already been calculated and is given by formula (8.A.8):

$$\frac{1}{\tau_{\text{radiative}}} = \frac{1}{W}\frac{p_0^2\omega_{\text{atom}}^4}{12\pi\varepsilon_0 c^3}. \tag{8.A.10.a}$$

To obtain a value of τ_{rad} we need to know W; for no good reason, except that the result of the calculation is correct, we will replace W by $h\nu$:

$$\frac{1}{\tau_{\text{radiative}}} = \frac{p_0^2}{6h\varepsilon c^3}\omega_{\text{atom}}^3. \tag{8.A.10.b}$$

Our demonstration is not really satisfactory, since we treat in a classical way the spontaneous emission (see Chapter 9), which is typically a quantum process. Despite its impure origin, formula (8.A.10.b) is valid and verified by the experiment; a more rigorous demonstration can be given on a quantum basis. We have chosen this kind of presentation because it insists on the physical meaning of electric dipole radiation.

Order of Magnitude of the Radiative Lifetime

To do numerical applications of (8.A.10.b), we need an order of magnitude of p_0, we will make the product of the elementary charge $e = 1.6 \times 10^{-19}$ C by an order of magnitude of the diameter of an atom ($1\,\text{Å} = 10^{-10}\,\text{m}$) $\rightarrow$ $p_0 \approx 10^{-29}$ Cb.m. The time constant $\tau_{\text{radiative}}$ is also called the *relaxation time* of the transition.

Wavelength	1 m	1 cm	100 μm	10 μm	1 μm	0.1 μm	1 Å
Frequency	300 MHz	30 GHz	3×10^{12} Hz	3×10^{13} Hz	3×10^{14} Hz	3×10^{15} Hz	3×10^{18} Hz
$\tau_{\text{radiative}}$ Lifetime	10^{12} s millennium	10^{6} s one day	1 s	10^{-3} s	10^{-6} s μs	10^{-9} s ns	10^{-18} s

Orders of Magnitude of the Relaxation Times of Fully Allowed Transitions, $p_0 \approx 10^{-29}$ Cb.m

The above table gives some orders of magnitude of the relaxation times. Very long (a few days or more) relaxation times are not really meaningful: they just indicate that, in such cases, the system will find other relaxation processes (collisions with other atoms or with the walls of the tank, in the case of a gas; collision with phonons in a crystal).

As far as optical transitions are concerned the relaxation times are usually in the nanosecond-microsecond range. Radiative lifetimes can be measured; experimental values, τ_{measured}, are always larger than the theoretical value given by (8.A.10.b). The value f of the ratio of the calculated to the measured lifetime is smaller than one:

$$0 \leq f = \frac{\tau_{\text{radiative}}}{\tau_{\text{measured}}} \leq 1.$$

It can be shown that the parameter f is identical to the *oscillator strength* of formulas (8.12).

- If $f = 0$, the transition is said to be *forbidden*, the lifetime is then infinite.
- If $f = 1$, the transition is said to be *fully allowed*, the lifetime is equal to the calculated value.
- If $0 < f < 1$, the transition is said to be *partly allowed* (or *partly forbidden*).

The lifetime was calculated in the case of an isolated dipole that was radiating in the open space. Inside an atom, an electron belongs to the electron

cloud, which constitutes a type of electrical screening. We can give the following analogy of a dipole that is inside a closed box. At first the walls of the box are perfect conductors, the radiation that is produced by the dipole is reflected by the walls and stored: the lifetime is infinite, but an external observer is not aware that there is an oscillating dipole inside. Let us suppose now that small holes are drilled through the walls; only a part of the radiation will be diffracted outside while the remaining part is sent back to the dipole.

A transition associated to an electron belonging to an outer shell is fully allowed, this is the case of the emission of the well-known D-light by sodium atoms (there is only one peripheral electron); the lifetime of the excited level is of a few nanoseconds.

A transition associated to an electron belonging to an inner shell is only partly allowed, such cases are often met with in rare earth atoms and transition atoms of the Mendeleev chart. The lifetimes are then measured in milliseconds ($f \approx 10^{-3}$), the excited level is then said to be *metastable* (in ancient Greek, *meta* = almost). A famous example is given by chromium ions Cr^{3+}, which have been selected to obtain the first laser emission ($0.6943\,\mu$m) because of the existence of a metastable level, with a lifetime of three milliseconds.

Absorption and Stimulated Emission Are the Same Kind of Phenomena

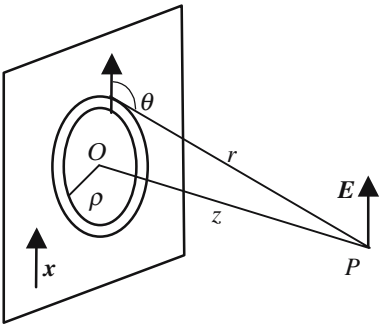
Absorption and stimulated emission will be introduced on a quantum basis in Section 9.2, however, contrary to spontaneous emission, they can be classically interpreted. We consider a dipole oscillating at some frequency ω and receiving an electromagnetic wave of the same frequency. The dipole interacts with the electric alternating field of the wave, the difference between the phase of the field and the phase of the motion of the dipole plays a key role; it is interesting to understand what happens when $\varphi = \pi$ or when $\varphi = 0$. When the motion and the field variation are in phase, the action of the field increases the amplitude of the dipole vibrations, which in turn gains more energy; on the contrary, if φ is equal to π, the dipole loses energy.

Let us now see what happens on the field side: stimulated emission corresponds to the case of phase concordance where the two fields are mutually reinforced by positive interference; absorption corresponds to the case of opposite phases and of attenuation by destructive interference.

8.A.4. Field Radiated by a Planar Distribution of Oscillating Dipoles

Richard Feynman in his *Lectures on Physics* proposed the following calculation. We refer to Figure 8.A.4, electrical charges are spread on a plane with a

surface density η that keeps a constant density η_0 inside a circle of radius R_0, and then rapidly decreases outside the circle. The distribution admits the symmetry of revolution around the axis of the circle. All the charges are vibrating in phase, with the same frequency ω and the same amplitude x_0, the direction of vibration is orthogonal to the plane. We want to calculate the electromagnetic field that is radiated at a point P located on the axis of the circle at a distance z that is far larger than R_0.



$$OP = z,$$

$$r^2 = \rho^2 + z^2 \quad \Rightarrow \quad r\,dr = \rho\,d\rho,$$

$$\theta \approx \pi/2 \quad \rightarrow \quad \sin\theta \approx 1,$$

$$\boldsymbol{dE} = \boldsymbol{x}\frac{q\omega^2 x_0}{4\pi\varepsilon_0 c^2}\frac{\eta}{r}e^{j\omega(t-r/c)}2\pi\rho\,d\rho.$$

Figure 8.A.4. The plane supports a distribution of oscillating dipoles; the vibrations are orthogonal to the plane. The elementary electric fields generated by the different dipoles are parallel to the unit vector $\boldsymbol{x}$.

The electric field created by the dipoles is given by the following integral:

$$E = \boldsymbol{x}\frac{q\omega^2 x_0}{2\varepsilon_0 c^2}e^{j\omega t}\int_z^\infty \eta e^{-j\omega r/c}dr.$$

The presence of the term $e^{-j\infty}$ is very unpleasant, this is the reason why we have chosen that η goes to zero when the distance to the center of the circle becomes infinite. The field is then given by

$$\boldsymbol{E} = -\boldsymbol{x}j\omega\frac{\eta_0 q x_0}{2\varepsilon_0 c}e^{j\omega(t-z/c)}. \tag{8.A.11}$$

Annex 8.B

The Kramers-Krönig Formula

8.B.1. Demonstration of the Kramers-Krönig Formula

The demonstration of this formula requires a rather sophisticated knowledge of mathematics and, more specifically, about Riemann and Cauchy integrals. A presentation can be found in the book *Solid State Physics* by C. Kittel, here we will just emphasize the physical articulations of the demonstration. By the way, we would like to underline the specific beauty of the argument which put the Principle of Causality on a mathematical basis and which also establishes a well-known experimental result, according to which blue light is more deviated than red light by a prism.

Let $\boldsymbol{P}$ be the polarization taken by a piece of material when submitted to an electric field $\boldsymbol{E}$. If the field varies with time, so does the polarization. A sine variation of the field will create a sine variation of the polarization. The complex vector $\overline{\boldsymbol{P}}$ is proportional to $\overline{\boldsymbol{E}}$: the field plays the role of an excitation, the polarization being the response,

$$\begin{gathered}\overline{\boldsymbol{E}}e^{-j\omega t} \rightarrow \overline{\boldsymbol{P}}e^{-j\omega t},\\ \overline{\boldsymbol{P}}e^{j\omega t} = \chi(\omega)\overline{\boldsymbol{E}}e^{j\omega t} = [\chi'(\omega) - j\chi''(\omega)]\overline{\boldsymbol{E}}e^{j\omega t}.\end{gathered} \tag{8.B.1}$$

If the excitation is no longer sinusoidal, but follows some other law of variation versus time, the situation is more complicated and the Fourier transform method should be used. We will now consider the case where the excitation is a Dirac pulse,

$$\begin{gathered}\boldsymbol{E}(t) = \boldsymbol{E}_0\delta(t) = \frac{\boldsymbol{E}_0}{2\pi}\int_{-\infty}^{+\infty} e^{j\omega t}\, d\omega,\\ \boldsymbol{P}(t) = \int_{-\infty}^{+\infty} \boldsymbol{P}(\omega)e^{j\omega t}\, d\omega = \varepsilon_0\boldsymbol{E}_0\int_{-\infty}^{+\infty} \chi(\omega)e^{j\omega t}\, d\omega.\end{gathered} \tag{8.B.2}$$

Formula (8.B.2) is a well-known result; the harmonic susceptibility $\chi(\omega)$ (ratio response/excitation in the harmonic case), is the inverse Fourier transform of the time response to a Dirac pulse,

$$\chi(\omega) = \frac{1}{\varepsilon_0} \int_{-\infty}^{+\infty} P(t) e^{-j\omega t}\, dt = \frac{1}{\varepsilon_0} \int_{0}^{+\infty} P(t) e^{-j\omega t}\, dt. \tag{8.B.3}$$

Since the response cannot exist prior to excitation, $P(t)$ is identically equal to zero when $t < 0$. The domain of integration of the integral of the Fourier transform should be limited to $[0, \infty]$.

It can be established mathematically that formula (8.B.3) implies that the real and imaginary parts of the harmonic susceptibility are *Hilbert transforms* of one another. Under such conditions the function $\chi(\omega)$ has the following properties:

- The poles (values of the frequency that make the function infinite) are all above the real axis.
- $\chi(\omega) \to 0$ when $\omega \to \infty$.
- $\chi'(\omega)$ is an even function and $\chi''(\omega)$ is an odd function of the real variable ω.
- $$\left.\begin{aligned} \chi'(\omega) &= \frac{2}{\pi} PP \int_0^{+\infty} \frac{\omega' \chi''(\omega')}{(\omega'^2 - \omega^2)} d\omega', \\ \chi''(\omega) &= -\frac{2\omega}{\pi} PP \int_0^{+\infty} \frac{\chi'(\omega')}{(\omega'^2 - \omega^2)} d\omega', \end{aligned}\right\}$$ the Kramers-Krönig formula.

The integrals raise some problems, since $\omega = \omega'$ is a pole; this is the reason why the Principal Parts (PP) have been introduced in the Kramers-Krönig formulas.

It's a good exercise to examine how the Kramers-Krönig formulas can be applied to formulas (8.15.a) and (8.15.b). As an example, consider the case of an atom with only one absorption band and derive the expression of $\chi''(\omega)$ from the expression of $\chi'(\omega)$; in other words, start from

$$\chi'(\omega) = \frac{Nq^2}{2\varepsilon_0 m \omega_0} \frac{(\omega_0 - \omega)}{\left[(\omega_0 - \omega)^2 + \gamma_k^2/4\right]},$$

and obtain

$$\chi''(\omega) = \frac{Nq^2 \nu}{2\varepsilon_0 m \omega_0} \frac{\gamma}{\left[(\omega_0 - \omega)^2 + \gamma^2/4\right]}.$$

8.B.2. Normal Dispersion

We consider a piece of material that is transparent inside a window limited by the two frequencies ω_1 and ω_2, the absorption being equal to zero at any frequency between ω_1 and ω_2,

$$\omega_1 < \omega < \omega_2 \quad \rightarrow \quad \chi''(\omega) = 0 \quad \rightarrow \quad \int_{\omega_1}^{\omega_2} \frac{\omega' \chi''(\omega')}{(\omega'^2 - \omega^2)} d\omega' = 0.$$

Anybody able to juggle with principal parts of integrals will easily establish that

$$\chi'(\omega) = \frac{2}{\pi} \int_0^{\omega_1} \frac{\omega' \chi''(\omega')}{(\omega'^2 - \omega^2)} d\omega' + \frac{2}{\pi} \int_{\omega_2}^{\infty} \frac{\omega' \chi''(\omega')}{(\omega'^2 - \omega^2)} d\omega',$$

$$n = n' = \sqrt{\varepsilon_r'} = \sqrt{\chi'} \quad \rightarrow \quad \frac{dn}{d\omega} = \frac{1}{2\sqrt{\chi'}} \frac{d\chi'}{d\omega},$$

$$\frac{d\chi'}{d\omega} = \frac{2}{\pi} \int_0^{\omega_1} \frac{2\omega\omega' \chi''(\omega')}{(\omega'^2 - \omega^2)^2} d\omega' + \frac{2}{\pi} \int_{\omega_2}^{\infty} \frac{2\omega\omega' \chi''(\omega')}{(\omega'^2 - \omega^2)^2} d\omega'. \tag{8.B.4}$$

The arguments of the integrals in (8.B.4) being positive, the derivatives $d\chi'/d\omega$ and $dn/d\omega$ are also positive and, inside a transparency band, the index of refraction increases with the frequency.