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Vectorial Propagation of Light

Maxwell's equations are the basis of all optical studies. In vacuum the equations can be stripped to a pure form where the wave motion is most easily described. Moreover, as the equations in vacuum are linear, each Fourier component of a wave can be individually studied and subsequently superimposed to construct a composite wavefront or ray bundle. When the electromagnetic wave propagates through media, additional terms are added to Maxwell's equations to account for the interaction. These terms come in as constitutive laws of the media. Constitutive laws can encompass lossy, charged, dielectric, nonlinear, or relativistic media. There is almost no end to the studies on optical interactions already undertaken over the last several hundred years.

The purpose of this chapter and that of Chapters 2 and 3 is to derive the necessary governing equations for studies of birefringent media, birefringent components, and birefringent effects in optical fiber. This chapter exclusively deals with Maxwell's equations in vacuum. The classical description of polarization motion and the degree of polarization is emphasized. Chapter 2 presents a modern description of polarization that adopts well-developed mathematical formalisms from quantum mechanics to polarization studies. Chapter 3 adds interaction terms to Maxwell's equations to describe optical propagation through birefringent linear dielectrics.

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1.1 Maxwell's Equations and Free-Space Solutions

The four vectorial Maxwell's equations are

Faraday's law: $$\nabla \times \mathbf{E}(\mathbf{r},t) = -\frac{\partial}{\partial t}\mu_o \mathbf{H}(\mathbf{r},t) - \frac{\partial}{\partial t}\mu_o \mathbf{M}(\mathbf{r},t)$$

Ampère's law: $$\nabla \times \mathbf{H}(\mathbf{r},t) = \varepsilon_o \frac{\partial}{\partial t}\mathbf{E}(\mathbf{r},t) + \frac{\partial}{\partial t}\mathbf{P}(\mathbf{r},t) + \mathbf{J}(\mathbf{r},t)$$

Gauss's electric law: $$\nabla \cdot \varepsilon_o \mathbf{E}(\mathbf{r},t) = -\nabla \cdot \mathbf{P}(\mathbf{r},t) + \rho_f(\mathbf{r},t)$$

Gauss's magnetic law: $$\nabla \cdot \mu_o \mathbf{H}(\mathbf{r},t) = -\nabla \cdot \mu_o \mathbf{M}(\mathbf{r},t)$$

where the vector quantities $\mathbf{E}$, $\mathbf{H}$, $\mathbf{P}$, $\mathbf{M}$, $\mathbf{J}$, and ρ_f are real functions of time t and the three-dimensional spatial vector $\mathbf{r}$. These vector quantities are

$\mathbf{E}(\mathbf{r},t)$:	electric field strength	(V/m)
$\mathbf{H}(\mathbf{r},t)$:	magnetic field strength	(A/m)
$\mathbf{P}(\mathbf{r},t)$:	polarization density	(C/m^2)
$\mathbf{M}(\mathbf{r},t)$:	magnetization density	(A/m)
$\mathbf{J}(\mathbf{r},t)$:	current density	(A/m^3)
$\rho_f(\mathbf{r},t)$:	electric charge density	(C/m^3)

where V is volts, A is amperes, and C is coulombs. The free electric charge density ρ_f is distinguished from the bound charge density ρ_b as the bound density is the generator of the polarization density vector $\mathbf{P}$.

The physical constants ε_o and μ_o are the permittivity and permeability of vacuum, respectively. The values and units are [8]

$$\varepsilon_o \simeq 8.854187817 \times 10^{-12} \quad (\mathrm{F/m})$$

$$\mu_o = 4\pi \times 10^{-7} \quad (\mathrm{H/m})$$

where F is Farads and H is Henries.

Maxwell's equations completely describe the propagation and spatial extent of electromagnetic waves in free-space and in any medium. Faraday's law states that the curl of the electric field is generated by the temporal change of the magnetic field and the magnetization density vector. Ampère's law states that the curl of the magnetic field is generated by the temporal change of the electric field and the polarization density vector, as well as by currents of charged particles. Gauss's two laws govern the divergence of the electric and magnetic fields. The divergence is zero except in the presence of dipoles and electric charges.

It is customary when considering a restricted class of problems to eliminate various non-essential terms from the equations. As this text is predominantly focused on passive birefringent optical components, including interaction with fixed electric and magnetic fields, the current density $\mathbf{J}(\mathbf{r},t)$, and the free electric charge density $\rho_f(\mathbf{r},t)$ are set to zero. The reduced equations are

$$\nabla \times \mathbf{E}(\mathbf{r},t) = -\mu_o \frac{\partial}{\partial t}\mathbf{H}(\mathbf{r},t) - \mu_o \frac{\partial}{\partial t}\mathbf{M}(\mathbf{r},t) \ , \tag{1.1.1}$$

$$\nabla \times \mathbf{H}(\mathbf{r},t) = \varepsilon_o \frac{\partial}{\partial t}\mathbf{E}(\mathbf{r},t) + \frac{\partial}{\partial t}\mathbf{P}(\mathbf{r},t) \ , \tag{1.1.2}$$

$$\nabla \cdot \varepsilon_o \mathbf{E}(\mathbf{r},t) = -\nabla \cdot \mathbf{P}(\mathbf{r},t) \ , \tag{1.1.3}$$

$$\nabla \cdot \mathbf{H}(\mathbf{r},t) = -\nabla \cdot \mathbf{M}(\mathbf{r},t) \tag{1.1.4}$$

The field terms $\mathbf{E}$ and $\mathbf{H}$ are the two complementary components of an electromagnetic wave. The polarization and magnetization density vectors $\mathbf{P}$ and $\mathbf{M}$, respectively, are the means to describe the interaction of the electromagnetic field with matter. The density vectors $\mathbf{P}$ and $\mathbf{M}$ are related to the field quantities $\mathbf{E}$ and $\mathbf{H}$ by constitutive relations. The constitutive relations for various dielectric materials will be presented in detail in Chapter 3. This chapter details the most simple solutions to Maxwell's equations, the field solutions in vacuum.

In a vacuum, vectors $\mathbf{P}$ and $\mathbf{M}$ are zero. The wave equation for the electric field is derived by taking the curl of Faraday's law, substituting in Ampère's law, and reordering the temporal derivative $\frac{\partial}{\partial t}$ since it commutes with ∇. The wave equation for the magnetic field is similarly found. The electric-field wave equation is

$$\nabla \times \nabla \times \mathbf{E} = -\mu_o \varepsilon_o \frac{\partial^2}{\partial t^2}\mathbf{E} \tag{1.1.5}$$

Application of the vector identity $\nabla \times \nabla \times = \nabla\,(\nabla \cdot\ ) - \nabla^2\,(\ )$, and recognition that Gauss' law (1.1.3) dictates zero electric-field divergence in the absence of a fixed charge density, (1.1.5) simplifies to the Helmholtz wave equation:

$$\nabla^2 \mathbf{E} = \mu_o \varepsilon_o \frac{\partial^2}{\partial t^2}\mathbf{E} \tag{1.1.6}$$

The Helmholtz equation relates the spatial curvature of the electric field $\mathbf{E}(\mathbf{r},t)$ to its temporal second derivative, the factor of proportionality being $\mu_o\varepsilon_o$. The wave equation is otherwise invariant to spatial and temporal translation, spatial rotation, time reversal, and coordinate system selection. Moreover, the wave equation is linear in that

$$\nabla^2 (\mathbf{E}_1 + \mathbf{E}_2) = \mu_o \varepsilon_o \frac{\partial^2}{\partial t^2} (\mathbf{E}_1 + \mathbf{E}_2) \tag{1.1.7}$$

The linear property of the wave equation allows arbitrarily complex field distributions $\mathbf{E}$ to be constructed by Fourier synthesis or the method of superposition.

A monochromatic solution to (1.1.6) is

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_o \cos(\omega t - \mathbf{k} \cdot \mathbf{r}) \tag{1.1.8}$$

where $\mathbf{E}_o$, $\mathbf{k}$, and $\mathbf{r}$ are three-dimensional real-valued vectors and ω is the radial oscillatory frequency of the wave. $\mathbf{E}_o$ is the field amplitude at time and distance zero. $\mathbf{k}$ is the propagation vector of the field. The magnitude of $\mathbf{k}$, having units of inverse length, is the wavenumber $k = |\mathbf{k}|$. The monochromatic wave (1.1.8) is a travelling plane wave that propagates in the direction of $\mathbf{k}$ and oscillates at frequency ω.

When an underlying coordinate system is chosen so that the propagation direction of the wave $\mathbf{k}$ is coincident with a coordinate axis $\hat{r}$, i.e. $\mathbf{k} \parallel \hat{r}$, the spatial argument of (1.1.8) reduces to $\mathbf{k} \cdot \mathbf{r} = kr$. The monochromatic solution simplifies to $\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_o \cos(\omega t - kr)$. This is the equation of a plane wave whose phase fronts are constant in the plane perpendicular to $\hat{r}$ and whose amplitude is likewise constant in that plane. Picking a fixed phase position along the wavefront as it propagates along $\hat{r}$, $\omega t - kr =$ constant, it is found that the phase front travels at phase velocity $v_{\mathrm{ph}} = \omega/k$. The wavelength λ, as defined by the length along $\hat{r}$ between two adjacent field maxima, is $\lambda = 2\pi/k$.

Substitution of the monochromatic plane wave solution (1.1.8) into the wave equation (1.1.6) yields the dispersion relation that relates the wavenumber to the radial frequency:

$$k = \omega \sqrt{\mu_o \varepsilon_o} \tag{1.1.9}$$

As wave equation (1.1.6) is written for vacuum, the electromagnetic wavefront velocity is the speed of light, c. Using the dispersion relation, the speed of light is related to the free-space permittivity and permeability as

$$c = 1/\sqrt{\mu_o \varepsilon_o} \tag{1.1.10}$$

The speed of light in vacuum is [8]

$$c \simeq 299{,}792{,}458 \quad (\mathrm{m/s})$$

The wavenumber is therefore related to frequency and the speed of light via $k = \omega/c$.

The monochromatic wave (1.1.8) can be resolved into cartesian coordinates as follows. The field amplitude vector is resolved into three scalar components $\mathbf{E}_o = [E_x \ E_y \ E_z]^T$; the coordinate vector $\mathbf{r}$ is resolved as

$$\mathbf{r} = \hat{x}x + \hat{y}y + \hat{z}z \tag{1.1.11}$$

the wave vector $\mathbf{k}$ is resolved as

$$\mathbf{k} = \hat{x}k_x + \hat{y}k_y + \hat{z}k_z \tag{1.1.12}$$

A particular vector component of (1.1.8) takes associated elements from $\mathbf{E}$ and $\mathbf{k} \cdot \mathbf{r}$, e.g. $E(x,t) = E_x \cos(\omega t - k_x x)$. From (1.1.12), the wavenumber in cartesian coordinates is

$$k = \sqrt{k_x^2 + k_y^2 + k_z^2} \tag{1.1.13}$$

The monochromatic electric-field solution (1.1.8) has a magnetic field counterpart. Introduction of (1.1.8) into Faraday's law and solving for $\mathbf{H}$ by taking the time integral yields the magnetic field monochromatic solution

$$\mathbf{H}(\mathbf{r},t) = \sqrt{\frac{\varepsilon_o}{\mu_o}}\ \hat{k} \times \mathbf{E}_o \cos(\omega t - \mathbf{k} \cdot \mathbf{r}) \tag{1.1.14}$$

where the value of the wavenumber k has been pulled through by writing $\mathbf{k} = k\hat{k}$ and where $\hat{k}$ is a unit vector pointing in the direction of $\mathbf{k}$. The magnetic field has the same spatial and temporal dependence as the associated electric field. The scalar constant that relates the two field amplitudes is $\sqrt{\varepsilon_o/\mu_o}$. This physical constant is called the characteristic admittance of vacuum. The characteristic impedance, the inverse of the admittance, is approximately [8]

$$\sqrt{\frac{\mu_o}{\varepsilon_o}} \simeq 376.730313461 \quad \text{(ohms)}$$

Substitution of the field equations (1.1.8) and (1.1.14) into Maxwell's equations (1.1.1-1.1.4) for vacuum yields

$$\mathbf{k} \times \mathbf{E} = \omega\mu_o\mathbf{H} \tag{1.1.15a}$$

$$\mathbf{k} \times \mathbf{H} = -\omega\varepsilon_o\mathbf{E} \tag{1.1.15b}$$

$$\mathbf{k} \cdot \mathbf{E} = 0 \tag{1.1.15c}$$

$$\mathbf{k} \cdot \mathbf{H} = 0 \tag{1.1.15d}$$

These equations show the relation of the electric and magnetic field oscillations with respect to one another and with respect to the propagation direction $\mathbf{k}$. The divergence equations for the electric and magnetic fields (1.1.15c,d) show that there are no field components in the direction of propagation. That is, the longitudinal field components are zero; only transverse components exist. Both the electric and magnetic field oscillations are therefore perpendicular to $\mathbf{k}$. Moreover, the electric and magnetic field oscillations are mutually perpendicular. Calculation of $\mathbf{E} \cdot \mathbf{H}$ via (1.1.15a,b) results in

$$\mathbf{E} \cdot \mathbf{H} = -\frac{1}{\omega^2\mu_o\varepsilon_o}\ (\mathbf{k} \times \mathbf{H}) \cdot (\mathbf{k} \times \mathbf{E})$$

Application of the vector relation $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$ shows that $\mathbf{E} \cdot \mathbf{H} = 0$.

Combination of Faraday's and Ampère's laws has led to the wave equation (1.1.6), which in turn yielded a monochromatic plane-wave solution for both field components (1.1.8) and (1.1.14). Substitution of these field expressions into Maxwell's equations for vacuum leads to the conclusion that the vectors $(\mathbf{E}, \mathbf{H}, \mathbf{k})$ are mutually perpendicular. What remains is the calculation of energy flow of the propagating electromagnetic wave.

Poynting's theorem shows explicitly that conservation of energy is an immediate result of Maxwell's equations. The theorem states that the electromagnetic power flow into a volume must equal the rate of increase of stored electric and magnetic energy plus the total power dissipated. To arrive at the conservation equation, take the dot product of $\mathbf{H}$ with Faraday's law and the dot product of $\mathbf{E}$ with Ampère's law, and use the vector identity $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot \mathbf{a} \times \mathbf{b} - \mathbf{b} \cdot \mathbf{a} \times \mathbf{c}$. Poynting's energy conservation equation is

$$\begin{aligned} \nabla \cdot (\mathbf{E} \times \mathbf{H}) + \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon_o \mathbf{E} \cdot \mathbf{E}\right) + \frac{\partial}{\partial t}\left(\frac{1}{2}\mu_o \mathbf{H} \cdot \mathbf{H}\right) & + \\ \mathbf{E} \cdot \frac{\partial \mathbf{P}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mu_o \mathbf{M}}{\partial t} + \mathbf{E} \cdot \mathbf{J} & = 0 \end{aligned} \tag{1.1.16}$$

The Poynting theorem introduces a new vector quantity: $\mathbf{E} \times \mathbf{H}$. This is called the Poynting vector and represents the electromagnetic power flow density and has units of $(\mathrm{W/m^2})$. It is customary to represent the Poynting vector by the symbol $\mathbf{S}$:

$$\mathbf{S}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t) \tag{1.1.17}$$

The direction of $\mathbf{S}$ is the direction of power flow. The power flow direction is always orthogonal to both the $\mathbf{E}$ and $\mathbf{H}$ fields. Recalling Gauss' integral theorem,

$$\int_V \nabla \cdot \mathbf{F}\, dV = \oint_S \mathbf{F} \cdot d\mathbf{a}$$

the divergence of $\mathbf{F}$ enclosed by volume V equals the power flow through surface S out of the volume. Accordingly, $\nabla \cdot \mathbf{S}$ represents the power flow out of a differential volume. This power flow is balanced by the increase of stored electromagnetic energy W and by the power dissipated P_d. Symbolically [2],

$$\nabla \cdot \mathbf{S} + \frac{\partial W}{\partial t} + P_d = 0$$

The energy stored in the system is recoverable; the stored energy is reactive rather than resistive. The power dissipated is non-recoverable. In terms of the conservation equation, energy that can be grouped after the $\partial/\partial t$ operator is stored while the fixed power is dissipated. As an example, consider a volume V through which electric energy $W_e = 1/2\, \varepsilon_o \mathbf{E} \cdot \mathbf{E}$ flows. Denote the temporal profile as $W_e(t) = W_o f(t)$ where $f(t)$ is a positive, bounded scalar function of time and W_o is the maximum electric energy. The profile function is zero at $t = \pm\infty$. The time-integrated reactive power is

$$\int_{-\infty}^{+\infty} \frac{\partial}{\partial t} (W_o f(t))\, dt = 0$$

Integration over all time shows that no net power was left in volume V. Shown another way [1], for any intermediate time t_o, the energy into the volume V is

$$\int_{-\infty}^{t_o} \frac{\partial}{\partial t} (W_o f(t))\, dt = +W_o f(t_o)$$

After t_o, the energy into the volume V is

$$\int_{t_o}^{\infty} \frac{\partial}{\partial t} (W_o f(t))\, dt = -W_o f(t_o)$$

The energy that flows into V up to time t_o is fully recovered as $t \to +\infty$. However, consider the $\mathbf{E} \cdot \mathbf{J}$ term. Using Ohm's law relating current density to electric field, $\mathbf{J} = \sigma \mathbf{E}$, where σ is the charge density, the power dissipated is

$$\int_{-\infty}^{\infty} \sigma E^2(t) dt = \sigma E_o^2\, \mathcal{I}_p$$

where $\mathcal{I}_p$ is the integral of the square electric field $E^2(t)$ over all time and E_o is the maximum field amplitude assuming a bounded field-amplitude time profile. Only if $E(t) = 0$ for all time for finite σ will the dissipated power vanish, but this is the trivial case.

With this understanding of what constitutes stored energy and dissipated power, the stored energy present in Poynting's theorem is identified with

$$W = W_e + W_m = \frac{1}{2}\varepsilon_o \mathbf{E} \cdot \mathbf{E} + \frac{1}{2}\mu_o \mathbf{H} \cdot \mathbf{H} \tag{1.1.18}$$

and the power dissipated is identified with

$$P_d = \mathbf{E} \cdot \mathbf{J} \tag{1.1.19}$$

This leaves the remaining terms $\mathbf{E} \cdot \partial \mathbf{P}/\partial t$ and $\mathbf{H} \cdot \partial \mu_o \mathbf{M}/\partial t$ open to interpretation as energy storage terms or power dissipative terms. In general these two terms can be either; the particulars depend on the nature of the matter with which the electromagnetic field interacts. For example, in the case of linear dielectrics, $\mathbf{P} = \varepsilon_o \chi_e \mathbf{E}$, the dipole density follows the electric field instantaneously. The change of energy of the polarization density is then

$$\mathbf{E} \cdot \frac{\partial \mathbf{P}}{\partial t} = \frac{\partial}{\partial t} \left(\frac{1}{2} \varepsilon_o \chi_e \mathbf{E} \cdot \mathbf{E} \right)$$

where the energy stored in the polarization density is clearly reactive. If, on the other hand, the dipole density exhibits a delayed reaction to the electric field, as can be the case in highly resistive media, then one could write $d\mathbf{P}/dt = a\mathbf{E}$ where a is a scaling parameter [2]. Then,

$$\mathbf{E} \cdot \frac{\partial \mathbf{P}}{\partial t} = a\mathbf{E} \cdot \mathbf{E}$$

and the system is dissipative.

Earlier in this section the general plane-wave monochromatic field solutions in vacuum were found for both the electric and magnetic fields. The power flow density is found by $\mathbf{S} = \mathbf{E} \times \mathbf{H}$. Taking the cross of (1.1.8) and (1.1.14) yields

$$\mathbf{S}(\mathbf{r}, t) = \hat{k} \sqrt{\frac{\varepsilon_o}{\mu_o}}\, E_o^2 \cos^2(\omega t - \mathbf{k} \cdot \mathbf{r}) \tag{1.1.20}$$

The time average of the Poynting vector yields the average power flow of the electromagnetic field:

$$\langle \mathbf{S}(\mathbf{r}, t) \rangle = \frac{1}{2\pi} \int_0^{2\pi} \mathbf{S}(\mathbf{r}, t) d(\omega t) = \hat{k}\, \frac{1}{2} \sqrt{\frac{\varepsilon_o}{\mu_o}}\, E_o^2 \tag{1.1.21}$$

The time-average power flow of the electromagnetic field in vacuum is along the $\hat{k}$ direction, where $\hat{k}$ is perpendicular to planes of constant phase along the wave front. In the following chapters, dielectric anisotropy is introduced. The anisotropy will, in general, break the apparent identity that $\mathbf{S}$ and $\mathbf{k}$ run parallel to one another and instead induce the power flow and wave-front propagation directions to diverge.

1.2 The Vector and Scalar Potentials

In the absence of currents, free charges, and electric and magnetic dipoles, Maxwell's equations reduce to

$$\nabla \times \mathbf{E} = -\mu_o \frac{\partial}{\partial t} \mathbf{H} \tag{1.2.1a}$$

$$\nabla \cdot \mu_o \mathbf{H} = 0 \tag{1.2.1b}$$

$$\nabla \times \mathbf{H} = \varepsilon_o \frac{\partial}{\partial t} \mathbf{E} \tag{1.2.1c}$$

$$\nabla \cdot \varepsilon_o \mathbf{E} = 0 \tag{1.2.1d}$$

Under these circumstances, the magnetic and electric fields are solenoidal (having zero divergence). It is appealing to find the class of fields that *a priori* guarantee the solenoidal nature. Note the following vectors identities:

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0 \tag{1.2.2a}$$

$$\nabla \times (\nabla \psi) = 0 \tag{1.2.2b}$$

that is, the divergence of an arbitrary field curl $\nabla \times \mathbf{F}$ is solenoidal and the curl of an arbitrary potential gradient $\nabla \psi$ is irrotational.

The solenoidal nature of $\mu_o\mathbf{H}$ is guaranteed by equating it with the curl of the vector potential $\mathbf{A}$:

$$\mu_o\mathbf{H} = \nabla \times \mathbf{A} \tag{1.2.3}$$

Substitution of (1.2.3) into (1.2.1a) yields

$$\nabla \times \left(\mathbf{E} + \frac{\partial}{\partial t}\mathbf{A}\right) = 0 \tag{1.2.4}$$

Following (1.2.2b), (1.2.4) is guaranteed by defining $\mathbf{E}$ as

$$\mathbf{E} = -\nabla\Phi - \frac{\partial}{\partial t}\mathbf{A} \tag{1.2.5}$$

where Φ is the scalar potential. Maxwell's equations (1.2.1a,b) are guaranteed to be satisfied when $\mathbf{E}$ and $\mathbf{H}$ are expressed in terms of the vector potential $\mathbf{A}$ and scalar potential Φ as above. That said, $\mathbf{A}$ is not yet uniquely determined, as any field is defined by both its curl and divergence. The divergence of $\mathbf{A}$ has not yet been established. Without this, a shift of the vector potential by an arbitrary gradient, e.g. $\mathbf{A}' = \mathbf{A} + \nabla\psi$, would not change either $\mathbf{E}$ nor $\mathbf{H}$ but would indeed change Φ.

The divergence of $\mathbf{A}$ must be set with an eye toward guaranteeing the solutions to the remaining Maxwell's equations (1.2.1c,d). Substitution of (1.2.3, 1.2.5) into (1.2.1c) gives

$$\nabla \times (\nabla \times \mathbf{A}) = \mu_o\varepsilon_o\frac{\partial}{\partial t}\left(-\nabla\Phi - \frac{\partial}{\partial t}\mathbf{A}\right) \tag{1.2.6}$$

Expanding the double-curl on the left side and rearranging terms makes

$$\nabla^2\mathbf{A} = \mu_o\varepsilon_o\frac{\partial^2}{\partial t^2}\mathbf{A} + \nabla\left(\nabla\cdot\mathbf{A} + \mu_o\varepsilon_o\frac{\partial}{\partial t}\Phi\right) \tag{1.2.7}$$

The selection of the vector potential divergence is arbitrary since $\mathbf{E}$ and $\mathbf{H}$ are invariant. Therefore the most convenient choice is suitable. Accordingly, a wave equation for the vector potential can be established given the definition

$$\nabla\cdot\mathbf{A} + \mu_o\varepsilon_o\frac{\partial}{\partial t}\Phi = 0 \tag{1.2.8}$$

This choice is called the Lorentz gauge. This gauge in turn is used to generate a wave equation for the scalar potential through substitution into (1.2.1d). Together the wave equations are

$$\nabla^2\mathbf{A} = \mu_o\varepsilon_o\frac{\partial^2}{\partial t^2}\mathbf{A} \tag{1.2.9a}$$

$$\nabla^2\Phi = \mu_o\varepsilon_o\frac{\partial^2}{\partial t^2}\Phi \tag{1.2.9b}$$

In summary, the vector and scalar potentials are self-consistent fields that are constructed to satisfy all of Maxwell's equations by definition. The divergence and curl of the vector potential is completely specified, through which the link to the scalar potential is defined. The vector and scalar potentials provide an alternative means to find solutions to Maxwell's equations. In particular, plane wave solutions exemplified by (1.1.8) are highly convenient when the electromagnetic source is modelled infinitely far away and any dielectric or magnetic media are piece-wise uniform; Fourier techniques can be used to assemble a ray bundle that satisfies some boundary condition. In contrast, point sources generate nonuniform field patterns that cannot be modelled by plane waves. The vector and scalar potentials are necessary to find the requisite field solutions. As a particularly relevant example, Gaussian beam optics grants the adiabatic expansion of a ray bundle as fundamental. In this paraxial limit, the eigen-waves have a spherical phase curvature that is not present in a plane wave. In practice, which formalism is used, field solutions or vector potential solutions, is determined by the problem and the required degree of accuracy.

1.3 Time-Harmonic Solutions

The above developments of Maxwell's equations, monochromatic field solutions, and Poynting's theorem were performed in vector notation with only passing reference to an underlying coordinate system. Pure vector notation provides the most compact form of the equations, provides for direct comparison of the vector quantities, and allows for resolution onto any convenient coordinate system. In a analogous manner, complex exponential notation is like vector notation because there is no *a priori* selection to an underlying time reference. The use of cosine solutions in the previous section is certainly acceptable, but choice of $(\sin, \cos)$ requires selecting an underlying time reference from the beginning. To keep with real-valued functions at this point will lead to unnecessary analytic complexity when adding phases or multiplying frequencies. The equations and solutions of the preceding section will be recast into complex exponential notation to simplify the analytics.

One problem with complex exponential notation is that there is no customary sign of the argument. Physics texts usually use $\exp(-i\omega t)$, while engineering text usually use $\exp(j\omega t)$. Either selection is fine, as long as the derivations, particularly those regarding polarization, are consistent. This text chooses to use $\exp(j\omega t)$.

The operators $\Re\text{e}$ and $\Im\text{m}$ are used to translate between real functions and complex exponential functions. For a complex exponential $z = \exp(j\phi)$, the following relations are defined:

$$\Re\text{e}\{\text{z}\} = \frac{\text{z} + \text{z}^*}{2} \;, \quad \Im\text{m}\{\text{z}\} = \frac{\text{z} - \text{z}^*}{2} \tag{1.3.1}$$

and

$$z = \Re\mathrm{e}\{\mathrm{z}\} + \mathrm{j}\Im\mathrm{m}\{\mathrm{z}\} \tag{1.3.2}$$

where z^* is the complex conjugate of z.

The real-valued electric field is defined using complex exponential notation as

$$\mathbf{E}(\mathbf{r}, t) = \Re\mathrm{e}\left\{\mathbf{E}\ \mathrm{e}^{\mathrm{j}(\omega \mathrm{t} - \mathbf{k}\cdot\mathbf{r})}\right\} \tag{1.3.3}$$

where $\mathbf{E}$ is a complex vector. Moreover, $\mathbf{E}$ is written rather than $\mathbf{E}_o$ only for compactness of notation, but it is recognized that $\mathbf{E}$ is evaluated at $t = 0$ and $r = 0$. The real part of (1.3.3) is the same as (1.1.8). The remaining field, dipole, and current terms in Maxwell's equations undergo a similar substitution. Operations ∇ and $\frac{\partial}{\partial t}$ on the complex field undergo the following mapping:

$$\nabla \to -j\mathbf{k}$$

$$\frac{\partial}{\partial t} \to j\omega$$

Substitution of (1.3.3) and like terms into Faraday's law yields [7]

$$\Re\mathrm{e}\left\{-\mathrm{j}\left[\mathbf{k} \times \mathbf{E} - \omega\left(\mu_{\mathrm{o}}\mathbf{H} + \mu_{\mathrm{o}}\mathbf{M}\right)\right] \mathrm{e}^{\mathrm{j}(\omega \mathrm{t} - \mathbf{k}\cdot\mathbf{r})}\right\} = 0$$

This equation must hold true for all time and position. As the real part of the exponential term can take any value between $-1 \leq \Re\mathrm{e}\left(\exp(\mathrm{j}\phi)\right) \leq 1$, the remaining expression must equal zero. To summarize, Maxwell's equations in time-harmonic, plane-wave form are

$$\mathbf{k} \times \mathbf{E} = \omega\mu_o\left(\mathbf{H} + \mathbf{M}\right) \tag{1.3.4}$$

$$\mathbf{k} \times \mathbf{H} = -\omega\left(\varepsilon_o\mathbf{E} + \mathbf{P}\right) \tag{1.3.5}$$

$$\mathbf{k} \cdot \left(\varepsilon_o\mathbf{E} + \mathbf{P}\right) = 0 \tag{1.3.6}$$

$$\mathbf{k} \cdot \left(\mu_o\mathbf{H} + \mu_o\mathbf{M}\right) = 0 \tag{1.3.7}$$

where the fixed charge and current densities have been excluded. It is particularly relevant to remark that since the electric and magnetic Gaussian laws show zero divergence, (1.3.4 and 1.3.5) describe the field motion exclusively in the plane perpendicular to $\mathbf{k}$.

The Poynting theorem can likewise be recast into complex notation. The theorem is

$$\mathbf{k} \cdot \left(\mathbf{E} \times \mathbf{H}^*\right) = \omega\mu_o|H|^2 + \omega\varepsilon_o|E|^2 + \omega\mathbf{H}^* \cdot \mu_o\mathbf{M} + \omega\mathbf{E} \cdot \mathbf{P}^* \tag{1.3.8}$$

As long as there is no phase between $\mathbf{H}^*$ and $\mu_o\mathbf{M}$, and similarly between $\mathbf{E}$ and $\mathbf{P}^*$, then the power flow density experiences no gain or loss. However, a lead or lag of $\mathbf{M}$ to $\mathbf{H}^*$, or $\mathbf{P}^*$ to $\mathbf{E}$, introduces gain or loss into the system.

The complex Poynting vector is defined as

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}^* \tag{1.3.9}$$

and the time-average of $\mathbf{S}$ is found by

$$\langle \mathbf{S} \rangle = \frac{1}{2} \Re\mathrm{e} \{\mathbf{E} \times \mathbf{H}^*\} \tag{1.3.10}$$

The following identities are useful for time-harmonic calculations:

$$\langle \mathbf{a}(\mathbf{r}, t) \mathbf{b}^*(\mathbf{r}, t) \rangle = \frac{1}{2} \Re\mathrm{e} \{\mathbf{a}(\mathbf{r}) \mathbf{b}^*(\mathbf{r})\} \tag{1.3.11a}$$

$$\langle \mathbf{a}(\mathbf{r}, t) \cdot \mathbf{b}^*(\mathbf{r}, t) \rangle = \frac{1}{2} \Re\mathrm{e} \{\mathbf{a}(\mathbf{r}) \cdot \mathbf{b}^*(\mathbf{r})\} \tag{1.3.11b}$$

$$\langle \mathbf{a}(\mathbf{r}, t) \times \mathbf{b}^*(\mathbf{r}, t) \rangle = \frac{1}{2} \Re\mathrm{e} \{\mathbf{a}(\mathbf{r}) \times \mathbf{b}^*(\mathbf{r})\} \tag{1.3.11c}$$

1.4 Classical Description of Polarization

Thus far the study of the vectorial nature of light has shown that a planar electro-magnetic wave is a solution to Maxwell's equations in free space, and that the wave has a phase velocity, wavelength, and dispersion relation. Moreover, the relation between electric and magnetic fields and the power flow of the wave have been determined. This section is addressed to the evolution of the electric field in the plane perpendicular to the propagation direction. The motion of the electric field in this plane governs the polarization of the wave. Separate discussion of the magnetic wave motion is redundant as the magnetic field is immediately derived from the electric field using Faraday's law.

Consider a time-harmonic monochromatic plane wave (1.3.3) that travels in the $\hat{z}$ direction ($\mathbf{k} \cdot \mathbf{r} = kz$), Fig. 1.1. Since $\mathbf{k} \cdot \mathbf{E} = 0$ in vacuum, so there is no $\hat{z}$ component to the electric field. The most general form of the electric field vector is then

$$\mathbf{E}(z, t) = \begin{pmatrix} E_x \, \mathrm{e}^{j\phi_x} \\ E_y \, \mathrm{e}^{j\phi_y} \end{pmatrix} \mathrm{e}^{j(\omega t - kz)} \tag{1.4.1}$$

where $E_{x,y}$ are signed real numbers. The complex 2-row column vector is called the Jones polarization vector [5].

This plane wave propagates along the z-axis with wavelength $2\pi/k$ and phase velocity c. The two field components lie in the (x, y) plane and complete full cycles at rate ω. The polarization of the wave is governed by the electric-field evolution in the *xyBasis* plane. For convenience of notion but without loss of generality, $kz = \phi_x$. Using this reference plane and converting (1.4.1) to its real-valued counterpart, the electric field vector is

$$\mathbf{E}(x, y, t) = \hat{x} E_x \cos(\omega t) + \hat{y} E_y \cos(\omega t + \phi) \tag{1.4.2}$$

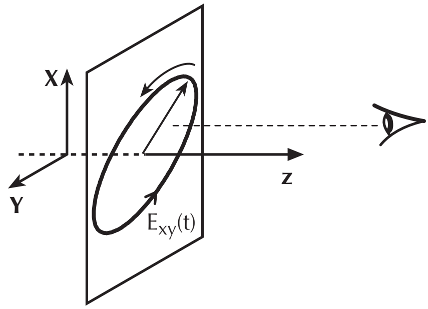

Fig. 1.1. In a vacuum, $\mathbf{k} \cdot \mathbf{E} = 0$, restricting the electric field to lie in the plane perpendicular to the propagation direction. Polarization is the motion of the electric field in the perpendicular plane.

where $\phi = \phi_y - \phi_x$. Equation (1.4.2) describes an ellipse in the plane perpendicular to $\hat{z}$. The convention used in this text to describe the state and handedness of the polarization ellipse is: the field is observed as it propagates towards the observer; that is, the observer faces in the $-\hat{z}$ direction, (see Fig. 1.1). The field is right-hand polarized when one's right-hand thumb points along $+z$ and one's figures curl in the direction of electric-field vector motion.

The elliptical equation is derived from (1.4.2) as follows. The field amplitudes as projected along the $\hat{x}$ and $\hat{y}$ directions are

$$x = E_x \cos(\omega t) \tag{1.4.3a}$$

$$y = E_y \cos(\omega t + \phi) \tag{1.4.3b}$$

Taking the square of the parametric equations, adding and absorbing terms by identification with $xy/E_x E_y$ yields the elliptical equation

$$\frac{x^2}{E_x^2} + \frac{y^2}{E_y^2} - \frac{2xy}{E_x E_y} \cos\phi = \sin^2\phi \tag{1.4.4}$$

There are three independent variables that govern the shape of the ellipse: E_x, E_y, and ϕ.

Figure 1.2 illustrates a general polarization ellipse resolved onto two coordinate systems. A general ellipse is one where there is no zero component in the (E_x, E_y, ϕ) triplet. In Fig. 1.2(a), $E_{x,y}$ mark the projections of the ellipse onto the (x, y) basis, and the angle χ is defined as $\tan\chi = E_y/E_x$ [4]. From the tangent relation between E_y and E_x, the Jones vector can be rewritten in normalized form:

$$\mathbf{E} = E_o \begin{pmatrix} \cos\chi \\ \sin\chi \, e^{j\phi} \end{pmatrix} \tag{1.4.5}$$

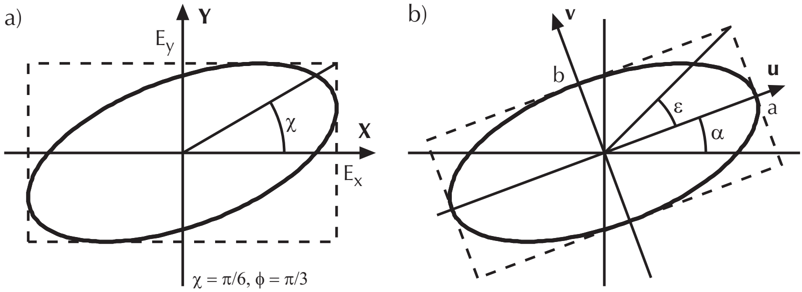

Fig. 1.2. Analysis of a general polarization ellipse onto the (x, y) and (u, v) coordinate systems. a) $E_{x,y}$ show maximum extent of elliptical motion on (x, y) basis. b) Same ellipse but where (u, v) basis is aligned to the major and minor elliptical axes. The angle between (x, y) and (u, v) is α.

where $E_o = \sqrt{E_x^2 + E_y^2}$ is the field amplitude irrespective of coordinate system. With this normalization, the state of polarization is described uniquely by the (χ, ϕ) pair of polarimetric parameters.

Now, as any ellipse has a major and minor axis, a coordinate system can be defined to align to these axes. Call this basis (u, v), Fig. 1.2(b). In the (u, v) basis the elliptical equation is

$$\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1 \tag{1.4.6}$$

where (a, b), the major and minor axes of the ellipse, are the projections onto the u and v axes, respectively. The parametric time-evolution equations that result in ellipse (1.4.6) are

$$u = a \cos \omega t \tag{1.4.7a}$$

$$v = b \sin \omega t \tag{1.4.7b}$$

As χ is defined as the tangent angle between E_y and E_x, ε is likewise defined as $\tan \varepsilon = b/a$. The ellipses described by (1.4.4) and (1.4.6) are related by a rotation in the plane through angle α. That is,

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{1.4.8}$$

Substituting the elliptical projections (1.4.3) and (1.4.7) into the above rotation, the angle of rotation α is

$$\tan 2\alpha = \tan 2\chi \cos \phi \tag{1.4.9}$$

To verify that the rotation was unitary, one can show that $a^2 + b^2 = E_x^2 + E_y^2$. An important conclusion is that while the (u, v) basis is the natural coordinate

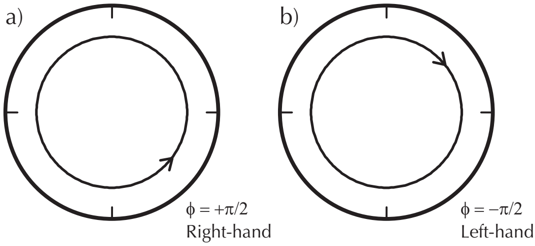

Fig. 1.3. Two states of circular polarization, counterclockwise (right-hand circular, or R) and clockwise (left-hand circular, or L). Right- and left-hand circular states are distinguished by the curl of one's fingers with the thumb pointing along the $+\hat{z}$ direction. Circular polarization exists when $\chi = \pm 45^o$ and $\phi = \pm\pi/2$. a) Counterclockwise (R) corresponds to $\phi = \pi/2$. b) Clockwise (L) corresponds to $\phi = -\pi/2$.

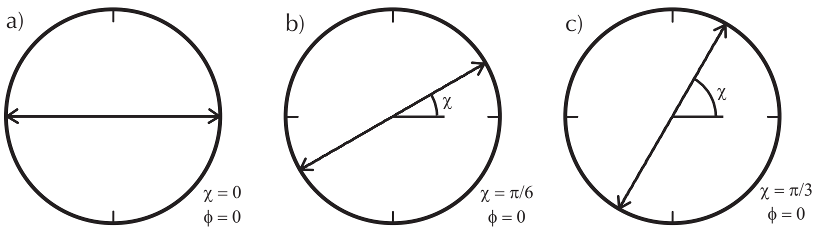

Fig. 1.4. Linear states of polarization exist when $\phi = m\pi$, where m is an integer. The orientation of the state is determined by χ, or alternatively by α. From a) to c), the value of α increases.

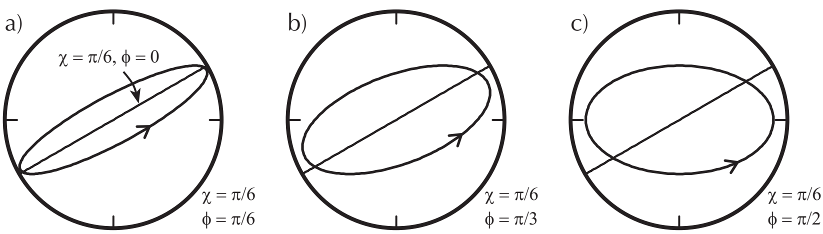

Fig. 1.5. Three elliptical polarization states. All three states have same value of χ. The phase difference ϕ increases: a) $\phi = \pi/6$, b) $\phi = \pi/3$, and c) $\phi = \pi/2$. Both χ and ϕ play a role in the orientation α of the ellipse, as governed by $\tan 2\alpha = \tan 2\chi \cos\phi$.

system for an ellipse having arbitrary rotation α, any unit ellipse may equally well be described on an arbitrary (x, y) basis by the (χ, ϕ) pair. The coordinate pairs (χ, ϕ) and (ε, α) are in one-to-one correspondence.

The parametric electric field described by (1.4.2) exhibits a handedness that depends on the sign of ϕ. For the range $-\pi \leq \phi < 0$, the evolution of the ellipse is in the clockwise (cw) direction and the handedness is left (L). For the range $0 < \phi \leq \pi$, the evolution is in the counterclockwise (ccw) direction and the handedness is right (R). The sense of the handedness is lost in elliptical equation (1.4.4) since $\cos\phi$ is an even function and $\sin^2\phi$ is positive definite. The same loss of handedness shows, however, that the shape of the ellipse is independent of the rotary sense.

There are three general categories of polarization state: circular, linear, and elliptical. Taken as a progression, circular is the most restrictive on the possible (χ, ϕ) values, linear is less restrictive, and elliptical places no restrictions on (χ, ϕ). In particular, circular polarization requires $\chi = \pm\pi/4$ and $\phi = \pm\pi/2$. Handedness is the only distinguishing property. When (χ, ϕ) have the same sign, the sense is R; when the signs are opposite the sense is L. Linear polarization lets χ take any value and requires $\phi = m\pi$, where m in an integer. Elliptical polarization includes circular and linear states as well as all other possible values of (χ, ϕ). Figures 1.3–1.5 provide examples of these three categories.

The polarization ellipse is completely described by the (χ, ϕ) pair. The question is how to determine these polarimetric parameters uniquely for an arbitrary state having arbitrary intensity. The following series of seven measurements will uniquely determine the state. The first measurement is for the overall time-averaged intensity. For a fixed polarization state

$$\mathbf{E} = \begin{pmatrix} E_x \\ E_y \ \mathrm{e}^{j\phi} \end{pmatrix} \tag{1.4.10}$$

where E_x and E_y are real, the time-averaged intensity is[1]

$$I_o = \frac{1}{2} \Re\mathrm{e}\,(\mathbf{E}^* \cdot \mathbf{E}) \tag{1.4.11}$$

$$= (E_x^2 + E_y^2)/2 \tag{1.4.12}$$

The remaining six measurements use a linear polarizer and, in two cases, a quarter-wave waveplate, to make the measurements. The projection matrix is a suitable model of a linear polarizer [10]

$$\mathcal{P} = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix} \tag{1.4.13}$$

[1] The time-average here is only over a few optical cycles. Partial polarization takes time-averages over longer periods.

The origin of this matrix is derived in Chapter 2. The angle θ is the angle of the polarizer to the horizontal axis. Any particular component intensity is calculated from $I_k \propto \mathbf{E}^\dagger \mathcal{P}(\theta)\mathbf{E}$. The first pair of measurements orient the polarizer in the $\hat{x}$ direction and $\hat{y}$ direction. The component intensities are

$$I_x = E_x^2/2 \tag{1.4.14a}$$

$$I_y = E_y^2/2 \tag{1.4.14b}$$

The second pair of measurements orient the polarizer in the $+45^o$ and -45^o directions. The component intensities are

$$I_{+45} = (E_x^2 + E_y^2)/4 + (E_x E_y/2)\cos\phi \tag{1.4.15a}$$

$$I_{-45} = (E_x^2 + E_y^2)/4 - (E_x E_y/2)\cos\phi \tag{1.4.15b}$$

One more measurement pair is necessary because handedness cannot be determined since $\cos\phi$ is an even function of ϕ. To complete the measurements, the optical beam is passed through a $+45°$-oriented quarter-wave waveplate and an $\hat{x}$- or $\hat{y}$-oriented polarizer so as to convert R and L hand circular polarizations to linear horizontal and vertical, respectively. The resulting intensities are

$$I_R = (E_x^2 + E_y^2)/4 + (E_x E_y/2)\sin\phi \tag{1.4.16a}$$

$$I_L = (E_x^2 + E_y^2)/4 - (E_x E_y/2)\sin\phi \tag{1.4.16b}$$

These seven measurements can be succinctly combined into four terms called Stokes parameters, which are defined by the equations

$$\begin{aligned}
S_0 &= I_x + I_y &&= (E_x^2 + E_y^2)/2 &&= \tfrac{1}{2}E_o^2 \\
S_1 &= I_x - I_y &&= (E_x^2 - E_y^2)/2 &&= \tfrac{1}{2}E_o^2 \cos 2\chi \\
S_2 &= I_{+45} - I_{-45} &&= E_x E_y \cos\phi &&= \tfrac{1}{2}E_o^2 \sin 2\chi \cos\phi \\
S_3 &= I_R - I_L &&= E_x E_y \sin\phi &&= \tfrac{1}{2}E_o^2 \sin 2\chi \sin\phi
\end{aligned} \tag{1.4.17}$$

From these equations the polarization coordinates (χ, ϕ) can be uniquely determined. Table 1.1 displays representative states in Jones and Stokes form.

1.4.1 Stokes Vectors, Jones and Muller Matrices

The Stokes vector $\mathbf{S}$ is defined by the projector construct (1.4.17). In general, one can write

$$\mathbf{S} = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} \tag{1.4.18}$$

The Stokes vector is the analogue to the Jones vector (1.4.5) on page 13. One must recognize that directly underlying the Jones vector are Maxwell's equations. The problem is that the Jones vector cannot be directly measured, but the Stokes vector can. The Jones vector is reconstructed from a Stokes vector to within a complex c constant by inverting (1.4.17):

$$\mathbf{E} = c \begin{pmatrix} \sqrt{\frac{1}{2}\left(1 + S_1/S_0\right)} \\ \sqrt{\frac{1}{2}\left(1 - S_1/S_0\right)}\ \exp\left(j \tan^{-1} S_3/S_2\right) \end{pmatrix} \tag{1.4.19}$$

Other than the undetermined complex constant c, there are three free variables in (1.4.19). A Jones vector, however, has four free variables: two amplitudes and two phases. The fourth free variable is the common phase of the two polarization components; this common phase is lost in the intensity measurements.

When light propagates through a medium, the interaction between medium and light can impart a change in the polarization state. In Stokes space, the change of state to $\mathbf{S}'$ from $\mathbf{S}$ is determined by the Mueller matrix $\mathbf{M}$. The general transformation is

$$\begin{pmatrix} S_0' \\ S_1' \\ S_2' \\ S_3' \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix} \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} \tag{1.4.20}$$

In matrix form one writes $\mathbf{S}' = \mathbf{MS}$. The Mueller matrix is a 4×4 matrix with real-valued entries. Polarimetric measurements find the Mueller matrix elements directly.

Underlying a Stokes-state transformation $\mathbf{M}$ is the Jones-state transformation $\mathbf{J}$. As with vectors, the Jones transformation matrix comes directly from Maxwell's equations; were it not for the natural advantages of polarimetric measurements the Mueller matrix would simply be a tautology. The Muller matrix is in any case the analogue to the Jones matrix. In Jones space, an output vector $\mathbf{E}'$ is related to the input vector $\mathbf{E}$ through

$$\mathbf{E}' = \mathbf{JE} \tag{1.4.21}$$

The Jones matrix $\mathbf{J}$ is a 2×2 matrix with complex-valued entries. The connection between Jones and Mueller matrices is derived using Pauli matrices (cf. §2.6.2). The Mueller matrix is derived from the Jones matrix via

$$\mathbf{M}_{i+1,j+1} = \frac{1}{2}\,\mathrm{Tr}\left(\mathbf{J}\sigma_j\,\mathbf{J}^\dagger \sigma_i\right) \tag{1.4.22}$$

where $i, j = 0, 1, 2$ or 3, σ_i is the i^{th} Pauli matrix, and Tr is the trace operator. The derivation of this expression is given in §2.6.2 starting on page 66.

Equation (1.4.22) is not invertible directly. However, R. C. Jones prescribes the way to reconstruct a Jones matrix from output Stokes vectors after three measurements [3, 6]. The three input states for the measurement are $\mathbf{S}_a = (1, 1, 0, 0)^T$, $\mathbf{S}_b = (1, -1, 0, 0)^T$, and $\mathbf{S}_c = (1, 0, 1, 0)^T$. Three output Jones vectors are constructed from the sequence:

$$\begin{pmatrix} \mathbf{S}_a \\ \mathbf{S}_b \\ \mathbf{S}_c \end{pmatrix} \xrightarrow{\mathbf{M}} \begin{pmatrix} \mathbf{S}'_a \\ \mathbf{S}'_b \\ \mathbf{S}'_c \end{pmatrix} \xrightarrow{\text{to Jones}} \begin{pmatrix} \mathbf{E}'_a \\ \mathbf{E}'_b \\ \mathbf{E}'_c \end{pmatrix} \tag{1.4.23}$$

From these three Jones vectors four complex ratios are calculated:

$$k_1 = E'_{xa}/E'_{ya}, \quad k_2 = E'_{xb}/E'_{yb} \quad k_3 = E'_{xc}/E'_{yc} \quad k_4 = \frac{k_3 - k_2}{k_1 - k_3} \tag{1.4.24}$$

To within a complex constant c, as before, the reconstructed Jones matrix is

$$\mathbf{J} = c \begin{pmatrix} k_1 k_4 & k_2 \\ k_4 & 1 \end{pmatrix} \tag{1.4.25}$$

Two classes of Jones matrices are particularly important for polarization studies: the Hermitian matrix and unitary matrix. Either matrix is written in the form

$$\mathbf{J} = \begin{pmatrix} a_0 + a_1 & a_2 - ja_3 \\ a_2 + ja_3 & a_0 - a_1 \end{pmatrix} \tag{1.4.26}$$

A Hermitian matrix represents a measurement of the polarization state and thus has real-valued eigenvalues. All four coefficients $a_{0,1,2,3}$ in (1.4.26) are real numbers. A unitary matrix represents a coordinate transformation of the Stokes vectors but imparts no loss or gain. Its eigenvalues are related through the matrix exponential, cf. §2.4.3. The Mueller equivalents to these matrices depend on the details, but the characteristic matrix forms are

$$\mathbf{J}_H \longrightarrow \mathbf{M}_H = \begin{pmatrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{pmatrix}, \quad \mathbf{J}_U \longrightarrow \mathbf{M}_U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet \end{pmatrix} \tag{1.4.27}$$

While a Hermitian matrix scatters energy to all elements of the Mueller matrix a unitary matrix keeps all of the light within the three spherical Stokes coordinates; the vector length S_0 remains unchanged. This characteristic form shows that $\mathbf{J}_U$ imparts only a rotation.

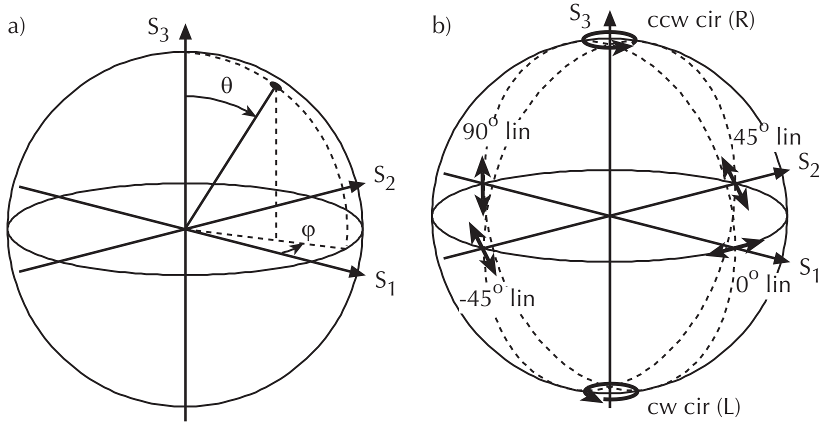

Fig. 1.6. Spherical representation of polarization states. a) The cartesian basis is (S_1, S_2, S_3). The equivalent spherical basis is (r, θ, φ). On a unit sphere, $r = 1$, so (θ, φ) coordinates uniquely determine position. b) Identification of particular polarization states on the Poincaré sphere. Along the equator lie linear states. At the north and south poles lie ccw (R) and cw (L) circular states. All remaining points are elliptical states. Orthogonal states are point pairs on opposite sides of the sphere connected by a cord that runs through the origin.

1.4.2 The Poincaré Sphere

Every possible polarization state can be represented on the surface of a unit sphere. The unit sphere is called the Poincaré sphere after H. Poincaré, its creator. A unit sphere is made by normalizing the three-directional Stokes components $S_{1,2,3}$ by the intensity component S_0. On a unit sphere, the declination and azimuth angles θ and φ describe any point on the surface. Referring to the polar coordinates illustrated in Fig. 1.6(a), the azimuth and declination angles are projected onto the (S_1, S_2, S_3) basis as

$$\begin{aligned} S_1 &= \sin\theta\cos\varphi \\ S_2 &= \sin\theta\sin\varphi \\ S_3 &= \cos\theta \end{aligned} \tag{1.4.28}$$

Associating spherical parameters to ellipse parameters $\theta = 2\varepsilon$ and $\varphi = 2\alpha$, the normalized Stokes components $S_{1,2,3}$ of (1.4.17) are related to the spherical coordinates as

$$\begin{aligned} S_1/S_0 &= \sin 2\varepsilon\cos 2\alpha &&= \cos 2\chi \\ S_2/S_0 &= \sin 2\varepsilon\cos 2\alpha &&= \sin 2\chi\cos\phi \\ S_3/S_0 &= \cos 2\varepsilon &&= \sin 2\chi\sin\phi \end{aligned} \tag{1.4.29}$$

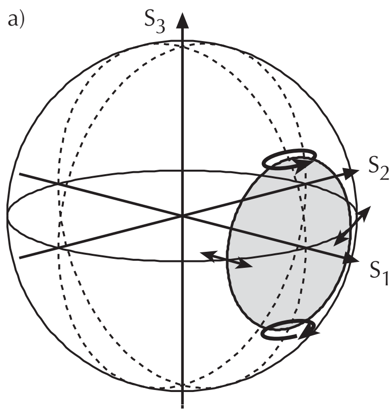

Fig. 1.7. Polarization contours. a) Contour of states for fixed χ and $-\pi \leq \phi \leq \pi$. The phase slips through a full revolution. This effect can be achieved physically by transmission through a waveplate. b) Contour of states for fixed ε and $-\pi/2 \leq \alpha \leq \pi/2$. The ellipse does a full rotation while maintaining its eccentricity. This effect can be achieved physically by transmission through an optically active waveplate.

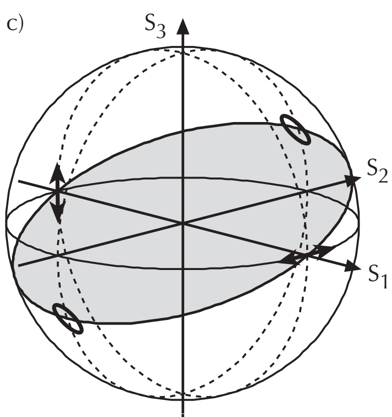

Fig. 1.7. Polarization contours. c) Contour of states for fixed ϕ and for $-\pi/2 \leq \chi \leq \pi/2$. χ determines the tilt of the plane. Any two orthogonal states lie on such a contour, the states being separated by 180°. d) Contour of states for fixed α and $-\pi \leq \varepsilon \leq \pi$. The eccentricity of the ellipse varies between linear and circular, but the pointing direction remains either vertical or horizontal.

Figure 1.6(b) illustrates the polarization states on the coordinate axes. Figure 1.7(a–d) illustrates various contours on the Poincaré sphere and their associations with $\varepsilon, \alpha, \chi$, and ϕ.

It is significant that the variables χ, ε, and α have a multiplier of two in (1.4.29) while ϕ does not. Physically, any full 2π phase slip of ϕ yields the identical polarization state; distinct optical phases within a 2π range correspond to distinct polarization states. In contrast, a π change in the χ, ε, and α parameters does not change the state. This is physically reasonable as an ellipse is preserved under 180° rotation, and $(E_x, E_y) \to (-E_x, -E_y)$ or $(a, b) \to (-a, -b)$ inversion. Jones space includes a built in degeneracy of elliptical parameters χ, ε, and α.

The spherical representation provides a geometric interpretation of the transformations that polarization states undergo when propagating through birefringent media. This representation will be used extensively throughout the text. There are, however, two drawbacks to the geometric interpretation. First, as the Stokes parameters are determined through measurements of intensity, only the polarization phase ϕ modulo 2π can be determined. In the study of polarization-mode dispersion, two orthogonally polarized waves can accrue thousands of 2π phase revolutions. As delay τ is defined as $\tau = \partial\phi/\partial\omega$, is it essential to track the total number of phase revolutions as well as any partial slip. Polarization-mode dispersion requires a modification to the Stokes calculus to treat the delay as well as the phase. Second, the polarization of a state by an arbitrarily oriented polarizer is difficult to picture in Stokes space. The projection due to the polarizer is more easily pictured in physical space. It is good practice to intuit a polarization state seamlessly in both Stokes and Jones space as a more robust understanding is achieved.

1.5 Partial Polarization

A wave is fully polarized when all component polarizations of a ray-bundle oscillate coherently. Such is the case with a laser. By contrast, "natural" light, such as light from the sun, is fully depolarized: the components of a ray-bundle are completely incoherent and the instantaneous polarization over a differential bandwidth can point in any direction on the Poincaré sphere. Partially polarized light can be "naturally" partially polarized in that some fraction of the ray-bundle is polarized and the remaining part "naturally" depolarized, or can be "pseudo" depolarized in that all components individually remain fully polarized but the polarization of the sum is not. The instantaneous polarization of pseudo-depolarized light touches a limited loci of points on the Poincaré sphere.

There are two ways to express partial polarization: the degree of polarization (DOP, denoted $\mathcal{D}$) and the Jones coherency matrix J. DOP is a scalar value between zero and one and can be expressed in terms of Stokes or Jones parameters. The Jones coherency matrix is derived from the dyadic form of the Jones vector and is used to trace depolarization through a system in Jones space. The coherency matrix is a necessary augmentation to Jones calculus because the 16 free variables of the Mueller matrix are enough to include depolarization directly, while that eight free variables of the Jones matrix do not provide enough freedom.

In terms of Stokes parameters, DOP is defined as

$$\mathcal{D} = \frac{\sqrt{\langle S_1 \rangle^2 + \langle S_2 \rangle^2 + \langle S_3 \rangle^2}}{\langle S_0 \rangle} \tag{1.5.1}$$

where the time averages are given by

$$\langle S(t) \rangle = \frac{1}{T} \int_0^T S(t) dt$$

The time average is taken over all time-varying quantities, i.e. ωt, $\chi(t)$, $\phi(t)$, etc. $\mathcal{D} = 1$ means that all waves that make up a ray bundle each have fully determined, time-invariant polarizations. $\mathcal{D} = 0$ means the polarimetric terms of the ray bundle have vanishing time averages, but the underlying cause, e.g. whether from incoherence or pseudo-depolarization, cannot be discerned using $\mathcal{D}$ alone. An intermediate value of $\mathcal{D}$ means that some of the optical power is polarized and the remaining power is not.

In terms of the coherency matrix, DOP is defined as

$$\mathcal{D} = \sqrt{1 - \frac{4 \ \det(J)}{\mathrm{Tr}(J)^2}} \tag{1.5.2}$$

The coherency matrix is defined by $J = \left\langle \mathbf{E}\mathbf{E}^\dagger \right\rangle$ [9], where

$$\mathbf{E}(t) = \begin{pmatrix} e_x(t) \\ e_y(t) \end{pmatrix}, \quad \text{and} \quad J = \begin{pmatrix} \langle e_x^* \, e_x \rangle & \left\langle e_x \, e_y^* \right\rangle \\ \langle e_x^* \, e_y \rangle & \left\langle e_y^* \, e_y \right\rangle \end{pmatrix} \tag{1.5.3}$$

and where (e_x, e_y) are complex numbers. Finally, the time-averaged Stokes parameters in terms of the coherency-matrix elements are

$$\begin{pmatrix} \langle S_0 \rangle \\ \langle S_1 \rangle \\ \langle S_2 \rangle \\ \langle S_3 \rangle \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -j & j \end{pmatrix} \begin{pmatrix} J_{xx} \\ J_{yy} \\ J_{xy} \\ J_{yx} \end{pmatrix} \tag{1.5.4}$$

Both $\mathcal{D}$ and J are inherently time-average measures. The integration period can affect the reported values. For instance, a monochromatic source that has a coherence time of 0.1 sec certainly produces polarized waves on time-scales $T << 0.1$ sec. However, polarization states separated by $T > 0.1$ sec are uncorrelated. A $\mathcal{D}$ measure taken over a long time scale would produce a sub-unity value, while a $\mathcal{D}$ measure over a short time scale would produce $\mathcal{D} \to 1$.

Both answers are technically correct and the issue reduces to what is a relevant time scale. That will depend on the application.

The following studies of partial polarization are grouped into ray bundles comprised of coherent, or polarized, components; incoherent, or depolarized, components; heterogeneous combinations of coherent and incoherent components; and pseudo-depolarized components. In all cases the ray-bundle components are collinear. In the following calculations, the electric-field spectrum is denoted as

$$\mathbf{E}(\omega) = E_o G(\omega) \sum_n \hat{p}_n(\omega) \tag{1.5.5}$$

where $G(\omega)$ is the spectral profile, E_o is complex, and $\hat{p}_n(\omega)$ is the n^{th} polarization at ω. The time-dependent field $\mathbf{E}(t)$ is the inverse Fourier transform of $\mathbf{E}(\omega)$:

$$\mathbf{E}(t) = \sum_n \int E_o G(\omega) \hat{p}_n(\omega) \mathrm{e}^{j\omega t} d\omega \tag{1.5.6}$$

1.5.1 Coherently Polarized Waves

The common feature of the four cases studied below is that the polarization of each component is time-invariant and independent of frequency. The study begins with a single monochromatic wave and generalizes to narrowband ray bundles having either discrete or continuous spectra. The studies show that for coherently polarized waves, only pseudo-depolarization can reduce the degree of polarization below unity.

A Monochromatic Polarized Wave

The simplest case is a single monochromatic polarized plane wave. The field spectrum is

$$\mathbf{E}(\omega) = E_o \delta(\omega - \omega_o) \hat{p} \tag{1.5.7}$$

where $\delta(\omega - \omega_o)$ is the Dirac delta function centered at ω_o. In the time domain, the plane wave is

$$\mathbf{E}(t) = E_o \, \mathrm{e}^{j\omega_o t} \begin{pmatrix} \cos\chi \\ \sin\chi \, \mathrm{e}^{j\phi} \end{pmatrix}$$

The corresponding Stokes parameters are

$$\langle S \rangle = |E_o|^2 \begin{pmatrix} 1 \\ \cos 2\chi \\ \sin 2\chi \cos\phi \\ \sin 2\chi \sin\phi \end{pmatrix} \tag{1.5.8}$$

As χ and ϕ are fixed in time, substitution of (1.5.8) into (1.5.1) yields $\mathcal{D} = 1$. The coherency matrix is

$$J = \begin{pmatrix} \cos^2\chi & \mathrm{e}^{-j\phi}\sin\chi\cos\chi \\ \mathrm{e}^{j\phi}\sin\chi\cos\chi & \sin^2\chi \end{pmatrix} \tag{1.5.9}$$

The polarization state of this wave is completely determined.

A Monochromatic Wave Having Multiple Polarizations

The spectrum of a ray bundle that comprises multiple monochromatic polarized waves of multiple polarization components is written as

$$\mathbf{E}(\omega) = \sum_n E_{on}\delta(\omega - \omega_o)\hat{p}_n \tag{1.5.10}$$

The time-domain field of the ray bundle is

$$\mathbf{E}(t) = \mathrm{e}^{j\omega_o t}\sum_n E_{on}\begin{pmatrix} \cos\chi_n \\ \sin\chi_n\ \mathrm{e}^{-j\phi_n} \end{pmatrix}$$

While the polarimetric parameters of the combined wave may be complicated, they do not vary in time. One can verify that

$$\langle S_1\rangle^2 + \langle S_2\rangle^2 + \langle S_3\rangle^2 = (e_x^* e_x + e_y^* e_y)^2$$

and thus $\mathcal{D} = 1$. A ray bundle that is constituted from multiple monochromatic coherent waves has a polarization state that is completely determined. The intensity of the ray bundle is calculated from $\langle S_1\rangle$, or

$$I_{\mathrm{coh}} = \langle S_0\rangle = \sum_n |E_{on}|^2 \tag{1.5.11}$$

where I_{coh} denotes the intensity of the coherent waves.

Narrowband Polarized Waves with Discrete Spectrum

Consider an extension of (1.5.7) where the spectrum comprises multiple frequency components, each component itself being polarized:

$$\mathbf{E}(\omega) = \sum_n E_{on}\delta(\omega - \omega_n)\hat{p}_n \tag{1.5.12}$$

In the time domain, this discretely polychromatic wave is

$$\mathbf{E}(t) = \sum_n \mathrm{e}^{j\omega_n t}E_{on}\begin{pmatrix} \cos\chi_n \\ \sin\chi_n\ \mathrm{e}^{j\phi_n} \end{pmatrix} \tag{1.5.13}$$

The polarimetric parameters χ_n and ϕ_n for each frequency component are fixed in time ($\mathcal{D}_n = 1$) and the frequencies ω_n are distinct. After summation, however, the composite polarimetric parameters do depend on time. In general this leads to $\mathcal{D}_{\text{total}} < 1$.

The depolarization is calculated as follows. Consider first the S_1 Stokes parameter:

$$\begin{aligned} S_1 &= e_x^* e_x - e_y^* e_y \\ &= \sum_m \mathrm{e}^{-j\omega_m t} E_{om}^* \cos\chi_m \sum_n \mathrm{e}^{j\omega_n t} E_{on} \cos\chi_n \\ &\qquad - \sum_m \mathrm{e}^{-j\omega_m t} E_{om}^* \sin\chi_m \mathrm{e}^{-j\phi_m} \sum_n \mathrm{e}^{j\omega_n t} E_{on} \sin\chi_n \mathrm{e}^{j\phi_n} \end{aligned}$$

The time averages on normalized components are

$$\left\langle \sum_m \mathrm{e}^{-j\omega_m t} \cos\chi_m \sum_n \mathrm{e}^{j\omega_n t} \cos\chi_n \right\rangle = \sum_n \cos^2\chi_n$$

and

$$\left\langle \sum_m \mathrm{e}^{-j\omega_m t} \sin\chi_m \mathrm{e}^{-j\phi_m} \sum_n \mathrm{e}^{j\omega_n t} \sin\chi_n \mathrm{e}^{j\phi_n} \right\rangle = \sum_n \sin^2\chi_n$$

where the time-average window is $T >> [\min(\omega_n - \omega_m)]^{-1}$. All cross terms are eliminated upon averaging, and the same holds for S_2 and S_3. In general the three time-averaged Stokes parameters are

$$\langle S_k \rangle = \sum_n \langle S_{kn} \rangle \tag{1.5.14}$$

Accordingly, the degree of polarization is

$$\mathcal{D} = \frac{\sqrt{\left(\sum_n \langle S_{1n}\rangle\right)^2 + \left(\sum_n \langle S_{2n}\rangle\right)^2 + \left(\sum_n \langle S_{3n}\rangle\right)^2}}{\sum_n \langle S_{0n}\rangle}$$

Since $\mathcal{D}_n$ of each component is unity, it follows that $\langle S_{0n}\rangle^2 = \langle S_{1n}\rangle^2 + \langle S_{2n}\rangle^2 + \langle S_{3n}\rangle^2$. By iterating the triangle inequality

$$|\mathbf{r}_1 + \mathbf{r}_2| \le |\mathbf{r}_1| + |\mathbf{r}_2|$$

one concludes that

$$\sqrt{\left(\sum_n \langle S_{1n}\rangle\right)^2 + \left(\sum_n \langle S_{2n}\rangle\right)^2 + \left(\sum_n \langle S_{3n}\rangle\right)^2} \le \sum_n \langle S_{0n}\rangle$$

Therefore, in general, the DOP for a discretely polychromatic ray bundle is

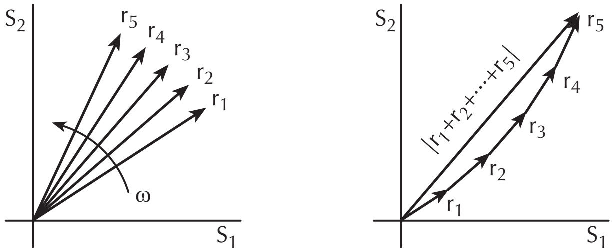

Fig. 1.8. Stokes vectors r_k in a plane. On the left, individual vector components: the vector direction is a function of frequency. On the right, the length of the vector sum is generally less than the arithmetic sum of the vector lengths.

$$\mathcal{D} = \frac{\sqrt{\left(\sum_n \langle S_{1n}\rangle\right)^2 + \left(\sum_n \langle S_{2n}\rangle\right)^2 + \left(\sum_n \langle S_{3n}\rangle\right)^2}}{I_{\text{coh}}} \leq 1 \tag{1.5.15}$$

where I_{coh} is given by (1.5.11). Equation (1.5.15) does provide some physical insight even though a specific expression is lacking. As Fig. 1.8 illustrates, when the Stokes vectors for the various frequencies are nearly aligned, then $\mathcal{D} \sim 1$. However, when the vector components are not aligned the overall DOP is reduced. Passage through a birefringent element can pseudo-depolarize this ray bundle (more detail is found in §1.5.3), but otherwise the addition of more coherent components in and of itself does not decrease the degree of polarization of the total.

A Narrowband Polarized Wave With Continuous Spectrum

A narrowband polarized wave is one where a modulation has been imprinted on a carrier. The broadening of the spectrum in this way does not entail a frequency-dependent polarization rotation. Accordingly, the spectrum is written as

$$\mathbf{E}(\omega) = E_o \left|G(\omega)\right| \mathrm{e}^{j\theta_G(\omega)}\hat{p} \tag{1.5.16}$$

where $G(\omega)$ is the modulated spectral profile. $G(\omega)$ is continuous for broadband modulation and discrete for harmonic modulation. The polarization direction is fixed along $\hat{p}$ and the profile amplitude is taken as a bound function which goes to zero outside a bandwidth of $\Delta\omega$. The time-domain electric field is

$$\mathbf{E}(t) = \hat{p}\int_{\Delta\omega} E_o \left|G(\omega)\right| \mathrm{e}^{j\theta_G(\omega)}\mathrm{e}^{j\omega t}d\omega$$

Consider first the ${e_x}^*e_x$ product:

$$\begin{aligned} {e_x}^*e_x \;&=\; \int_{\Delta\omega}\int_{\Delta\omega} \left|E_o\right|^2 \left|G(\omega_1)\right| \left|G(\omega_2)\right| \mathrm{e}^{j(\theta_G(\omega_2)-\theta_G(\omega_1))} \times \\ &\qquad \mathrm{e}^{j(\omega_2-\omega_1)t}\cos^2\chi \; d\omega_1 d\omega_2 \end{aligned}$$

Simplification comes with the time-average operation, where

$$\langle {e_x}^* e_x \rangle = \frac{1}{T} \int_0^T {e_x}^* e_x \, dt$$

generates a Dirac delta $\delta(\omega_2 - \omega_1)$ once the temporal integral is moved through to the $\exp j(\omega_2 - \omega_1)t$ term. Therefore,

$$\begin{aligned} \langle {e_x}^* e_x \rangle &= \int_{\Delta\omega} \int_{\Delta\omega} |E_o|^2 \, |G(\omega_1)| \, |G(\omega_2)| \, \mathrm{e}^{j(\theta_G(\omega_2) - \theta_G(\omega_1))} \times \\ &\qquad \cos^2 \chi \, \delta(\omega_2 - \omega_1) d\omega_1 d\omega_2 \\ &= |E_o|^2 \, \mathcal{I}_G \cos^2 \chi \end{aligned} \tag{1.5.17}$$

where the integral $\mathcal{I}_G$ is

$$\mathcal{I}_G = \int_{\Delta\omega} |G(\omega)|^2 \, d\omega \tag{1.5.18}$$

Following the same procedure,

$$\langle {e_y}^* e_y \rangle = |E_o|^2 \, \mathcal{I}_G \sin^2 \chi$$

and

$$\begin{aligned} \langle {e_x}^* e_y \rangle &= \langle e_x {e_y}^* \rangle^* \\ &= |E_o|^2 \, \mathcal{I}_G \sin \chi \cos \chi \, \mathrm{e}^{j\phi} \end{aligned}$$

The time-averaged Stokes parameters are

$$\langle S \rangle = |E_o|^2 \, \mathcal{I}_G \begin{pmatrix} 1 \\ \cos 2\chi \\ \sin 2\chi \cos \phi \\ \sin 2\chi \sin \phi \end{pmatrix} \tag{1.5.19}$$

and thus $\mathcal{D} = 1$. This derivation shows that line broadening due to modulation does not in itself alter the degree of polarization of the light. The light can be pseudo-depolarized, however. Contrary to a discrete spectrum, for a continuous spectrum $\mathcal{D} \to 0$ monotonically with increasing bandwidth-delay from the depolarizing element.

1.5.2 Incoherently Depolarized Waves

Incoherently depolarized waves are comprised of individual components having time-varying polarimetry parameters. Light from the sun or noise from an optical amplifier are examples of completely depolarized light. An exposed air-gap polarization-dependent delay line, used to generate differential-group delay, can have a time-dependent retardance with a fixed ellipsometric orientation. The DOP of this source depends on the orientation of the input state.

A Narrowband Incoherent Wave

An narrowband incoherent wave is one where the projection angle χ and/or the phase slip ϕ changes with time. The field amplitude may also change in time, but that impacts only the wave intensity rather than the polarization state. The time scale for $\chi(t)$ and $\phi(t)$ change is assumed to be significantly shorter than the integration time of the $\mathcal{D}$ measurement. Moreover, it is understood that $\phi(t)$ of a single wave is synonymous with frequency shift, which makes the wave technically narrowband rather than monochromatic; it is assumed that $\phi(t)$ changes slowly enough so that the line broadening is inconsequential. A narrowband, incoherent-wave spectrum is written as

$$\mathbf{E}(\omega) = E_o \delta(\omega - \omega_o)\hat{p}(B) \tag{1.5.20}$$

where B denotes a spectral bandwidth which is consistent with the integration time. The time-domain field is

$$\mathbf{E}(t) = E_o \, \mathrm{e}^{j\omega_o t} \begin{pmatrix} \cos\chi(t) \\ \sin\chi(t)\, \mathrm{e}^{j\phi(t)} \end{pmatrix} \tag{1.5.21}$$

The corresponding Stokes parameters are

$$S(t) = |E_o|^2 \begin{pmatrix} 1 \\ \cos 2\chi(t) \\ \sin 2\chi(t)\cos\phi(t) \\ \sin 2\chi(t)\sin\phi(t) \end{pmatrix} \tag{1.5.22}$$

Consider an exposed air-gap polarization-dependent delay line with a stable input polarization. The input polarization beam splitter projects the input light onto two orthogonal axes and delays one with respect to the other. For a stable input polarization, the projection is fixed in time: $\chi(t) = \chi_o$. The exposed delay arm, however, imparts a time-varying retardance. In this case, the time-averaged Stokes parameters are

$$\langle S \rangle = |E_o|^2 \begin{pmatrix} 1 \\ \cos 2\chi_o \\ 0 \\ 0 \end{pmatrix}$$

The degree of polarization is therefore

$$\mathcal{D} = |\cos 2\chi_o| \tag{1.5.23}$$

$\mathcal{D}$ can attain values $0 \le \mathcal{D} \le 1$. When $\chi_o = 0°$ all light travels in one arm or the other. Therefore $\mathcal{D} = 1$ as no relative phase shift is experienced. Alteratively, when $\chi_o = 45°$, the light is equally split between the two arms and $\mathcal{D} = 0$. One should be careful about the stability of air-gap polarization controllers.

Separately, consider the more general case where both χ and ϕ change in time. In this case $\langle S \rangle = [1\ 0\ 0\ 0]^T$ and $\mathcal{D} = 0$ over suitably long integration periods.

Multiple Narrowband Incoherent Waves

As an extension of (1.5.21), a ray bundle composed of multiple narrowband incoherent waves is written as

$$\mathbf{E}(t) = \mathrm{e}^{j\omega t} \sum_n E_{on} \left(\begin{array}{c} \cos\tilde{\chi}_n \\ \sin\tilde{\chi}_n \, \mathrm{e}^{j\tilde{\phi}_n} \end{array} \right) \tag{1.5.24}$$

where $\tilde{\chi}$ and $\tilde{\phi}$ denote random variables χ and ϕ in time. The distributions of $\tilde{\chi}$ and $\tilde{\phi}$ are uniform for each wave in the ray bundle. The compound polarimetric parameters depend on time, too, and the averages are found as follows. Consider first the S_1 term, where

$$\begin{aligned} S_1 &= e_x^* e_x - e_y^* e_y \\ &= \sum_m E_{om}^* \cos\tilde{\chi}_m \sum_n E_{on} \cos\tilde{\chi}_n \\ &\quad - \sum_m E_{om}^* \sin\tilde{\chi}_m \mathrm{e}^{-j\tilde{\phi}_m} \sum_n E_{on} \sin\tilde{\chi}_n \mathrm{e}^{j\tilde{\phi}_n} \end{aligned} \tag{1.5.25}$$

Now, since $\tilde{\chi}_m$ and $\tilde{\chi}_n$ are uncorrelated, only diagonal components of the product-of-sums are non-zero after time averaging. For any pair of indices,

$$\langle \cos\tilde{\chi}_m \cos\tilde{\chi}_n \rangle = \frac{1}{2}\, \delta_{m,n}$$

where $\delta_{m,n}$ is the Kronecker delta function defined by $\delta_{m,n} = 1$ if $m = n$ and $\delta_{m,n} = 0$ otherwise. The time averages over the sums are therefore

$$\left\langle \sum_m \cos\tilde{\chi}_m \sum_n \cos\tilde{\chi}_n \right\rangle = \frac{N}{2}$$

and

$$\left\langle \sum_m \sin\tilde{\chi}_m \mathrm{e}^{-j\tilde{\phi}_m} \sum_n \sin\tilde{\chi}_n \mathrm{e}^{j\tilde{\phi}_n} \right\rangle = \frac{N}{2}$$

where the time-average is "long enough" and the absence of the weighting coefficients is irrelevant in the limit. Therefore,

$$\langle S_1 \rangle \to 0$$

Now consider

$$\begin{aligned} \langle S_2 \rangle &= \langle e_x^* e_y + e_y^* e_x \rangle \\ &= \left\langle \sum_m \cos\tilde{\chi}_m \sum_n \sin\tilde{\chi}_n \mathrm{e}^{j\tilde{\phi}_n} \right\rangle + \left\langle \sum_m \sin\tilde{\chi}_m \mathrm{e}^{-j\tilde{\phi}_m} \sum_n \cos\tilde{\chi}_n \right\rangle \end{aligned}$$

Unlike (1.5.25), the time averages for both on- and off-diagonal components of $\langle S_2 \rangle$ are zero. Consequently,

$$\langle S_2 \rangle \to 0, \quad \text{and} \quad \langle S_3 \rangle \to 0$$

The only non-vanishing Stokes parameter is S_0, the total intensity. The time-average intensity I_{incoh} for an incoherently depolarized ray bundle is

$$I_{\text{incoh}} = \langle S_0 \rangle = \sum_n |E_{on}|^2 \tag{1.5.26}$$

and the degree of polarization is $\mathcal{D} = 0$.

A significant extension of the preceding derivation is that multiple incoherent waves need not be narrowband but can be discretely or continuously polychromatic. Relocation of the $\exp(j\omega t)$ term of (1.5.24) within the summations does not change the vanishing time-average nature of $\langle S_{1,2,3} \rangle$. However, polychromatic wave addition can relax the distribution property constraints of $\tilde{\chi}$ and $\tilde{\phi}$ to achieve $\mathcal{D} = 0$.

1.5.3 Pseudo-Depolarized Waves

Pseudo-depolarized waves are waves that start fully polarized and are then depolarized by passage through a birefringent crystal. This configuration is called a Lyot depolarizer. The depolarizer imparts a frequency-dependent polarization on the components of the input light. Unlike natural polarization where each light component uniformly covers the Poincaré sphere, pseudo-depolarized light retains a well-defined pointing direction for each polarization component; these directions vary with frequency.

Consider a single-crystal depolarizer oriented at 45° to a horizontally polarized input state. Denote $\tau = \Delta n L/c$, where Δn is the birefringence, L is the length, and c is the speed of light. The output polarization state is

$$\frac{e^{-j\omega\tau/2}}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{j\omega\tau} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-j\omega\tau/2} & \\ & e^{j\omega\tau/2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.5.27}$$

It is readily verified that $S_1 = 0$. The non-vanishing Stokes parameters are

$$S_2 = \cos\omega\tau\ , \quad S_3 = \sin\omega\tau$$

These parameters are time invariant, but the pointing direction of the Stokes vector changes with frequency. For this example, an arc along a line of longitude on the Poincaré sphere is traced, the subtended arc angle being $\omega\tau$.

More generally, consider the Jones matrix in (1.5.27) operating on a polarized narrowband wave having a continuous spectrum (1.5.16). The spectrum has a modified polarimetric parameter due to the $\exp(j\omega\tau)$ term. The time-domain field components are

$$
\begin{aligned}
e_x(t) &= E_o \cos\chi \int_{\Delta\omega} G(\omega)\mathrm{e}^{j\omega t} d\omega \\
e_y(t) &= E_o \sin\chi\, \mathrm{e}^{j\phi} \int_{\Delta\omega} G(\omega)\mathrm{e}^{j\omega\tau}\mathrm{e}^{j\omega t} d\omega
\end{aligned}
$$

Following the time-averaging procedure of (1.5.17), the off-diagonal components of J are

$$
\begin{aligned}
\langle {e_x}^* e_y \rangle &= \langle e_x {e_y}^* \rangle^* \\
&= |E_o|^2 \sin\chi \cos\chi\, \mathrm{e}^{j\phi} \mathcal{I}_G(\tau)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{I}_G(\tau) &= \int_{\Delta\omega} |G(\omega)|^2\, \mathrm{e}^{j\omega\tau} d\omega \\
&= \int_{\Delta\omega} |G(\omega)|^2 \cos(\omega\tau) d\omega + j \int_{\Delta\omega} |G(\omega)|^2 \sin(\omega\tau) d\omega
\end{aligned}
\tag{1.5.29}
$$

The diagonal components of J are

$$
\begin{aligned}
\langle {e_x}^* e_x \rangle = |E_o|^2\, \mathcal{I}_G(0) \cos^2\chi \\
\langle {e_y}^* e_y \rangle = |E_o|^2\, \mathcal{I}_G(0) \sin^2\chi
\end{aligned}
$$

Taking these factors into account, the Stokes parameters for a pseudo-depolarized narrowband wave are

$$
\langle S \rangle = |E_o|^2 \begin{pmatrix} \mathcal{I}_G(0) \\ \mathcal{I}_G(0)\cos 2\chi \\ |\mathcal{I}_G(\tau)| \sin 2\chi \cos\left(\phi + \angle\mathcal{I}_G(\tau)\right) \\ |\mathcal{I}_G(\tau)| \sin 2\chi \sin\left(\phi + \angle\mathcal{I}_G(\tau)\right) \end{pmatrix}
\tag{1.5.30}
$$

Since $|G(\omega)|^2$ is always positive, the sine and cosine integrands in (1.5.29) are the only sources able to decrease $\mathcal{I}_G(\tau)$, see Fig. 1.9. In the limit that $\tau \to 0$, the oscillatory terms are nearly stationary and $\mathcal{I}_G(\tau) \to \mathcal{I}_{G,\mathrm{max}}$. Conversely, when there is enough birefringent delay such that $\tau \gg \Delta\omega^{-1}$, the oscillatory terms vary rapidly, resulting in $\mathcal{I}_G(\tau) \to 0$. For a continuous spectrum, the DOP decreases monotonically with increasing delay-bandwidth product.

It is interesting to note that $\tau \gg \Delta\omega^{-1}$ is a necessary but not sufficient condition for a single-stage Lyot depolarizer to drive $\mathcal{D} \to 0$. If the input polarization is aligned to an eigenaxis of the crystal then there is no dispersion of the polarization vector over frequency. The DOP remains unity. The DOP is minimized when the input polarization is equally split between axes of the crystal. For this reason, two or more stages are generally used in a Lyot depolarizer.

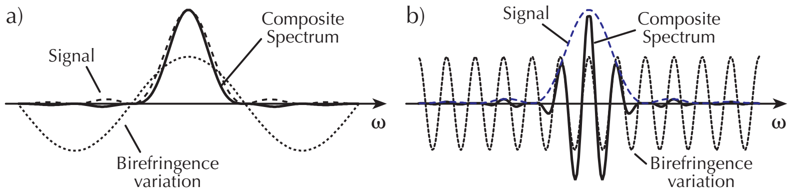

Fig. 1.9. Single-stage Lyot depolarizer impact on a continuous narrowband spectrum. a) Delay τ smaller than inverse signal bandwidth yields slow birefringence variation; the depolarizer has small effect on the integral $\mathcal{I}_G(\tau)$. b) Delay much larger than inverse signal bandwidth; $\mathcal{I}_G(\tau)$ is significantly smaller in this case. As τ increases, $\mathcal{D} \to 0$ monotonically.

In contrast to the continuous-spectrum case, consider a discrete spectrum described by

$$G(\omega) = \sum_n g_n \delta(\omega - \omega_n)$$

where amplitudes g_n decrease away from ω_o. In this case $\mathcal{I}_G(\tau)$ converts to

$$\mathcal{I}_G(\tau) = \sum_n g_n^2 \exp(j\omega_n \tau) \tag{1.5.31}$$

In contrast with the continuous wave, the integral $\mathcal{I}_G(\tau)$ does not monotonically decrease. Rather, the sum oscillates with a decreasing envelope as τ increases. The components of (1.5.31) are phasors (see Fig. 1.8), and the angle between adjacent phasors is determined by τ. As the phasors fan out for increasing τ eventually all even phasors point along $+1$ and all odd phasors point along -1. The sum is zero if the spectrum is symmetric. Subsequent doubling of τ points all phasors along $+1$. Such oscillation persists until the birefringence raps around within the linewidth of an individual spectral component.

1.5.4 A Heterogeneous Ray Bundle: Coherent and Incoherent Waves

The preceding sections have studied the DOP for coherent and incoherent ray bundles separately. Signals in a practical system such as a fiber-optic communication link are generally comprised of both coherent and incoherent terms. Coherent light comes from the laser source and incoherent light comes from both the noise of optical amplifiers and depolarization due to polarization-mode dispersion. The degree of polarization for such a heterogeneous mixture is

$$\mathcal{D} = \frac{\sqrt{\langle S_{1-\text{coh}} + S_{1-\text{incoh}} \rangle^2 + \langle S_{2-\text{coh}} + S_{2-\text{incoh}} \rangle^2 + \langle S_{3-\text{coh}} + S_{3-\text{incoh}} \rangle^2}}{\langle S_{0-\text{coh}} + S_{0-\text{incoh}} \rangle}$$

Since the incoherent components have vanishing time-averaged Stokes parameters other than S_0, only the coherent terms in the numerator survive. When there is no pseudo-depolarization in the system, the expression for the DOP is

$$\mathcal{D} = \frac{I_{\mathrm{coh}}}{I_{\mathrm{coh}} + I_{\mathrm{incoh}}} \tag{1.5.32}$$

but when the spectrum is pseudo-depolarized, cf. (1.5.15), the DOP expression is

$$\mathcal{D} \leq \frac{I_{\mathrm{coh}}}{I_{\mathrm{coh}} + I_{\mathrm{incoh}}} \tag{1.5.33}$$

For instance, when $I_{\mathrm{incoh}} = 0$, pseudo-depolarization can drive the DOP to $\mathcal{D} = 0$. One generally finds expression (1.5.32) in the literature, but the very real effect of polarization mode dispersion in fiber-optic systems leads to the more general expression (1.5.33).

Table 1.1. Polarization States in Equivalent Representations

Polarization state	Jones vector	Stokes vector	Coherency matrix
Linear $\hat{x}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
Linear $\hat{y}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
Linear at $\pm 45°$	$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ \pm 1 \\ 0 \end{pmatrix}$	$\frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$
Right-hand circular	$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ j \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$	$\frac{1}{2} \begin{pmatrix} 1 & -j \\ j & 1 \end{pmatrix}$
Left-hand circular	$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -j \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$	$\frac{1}{2} \begin{pmatrix} 1 & j \\ -j & 1 \end{pmatrix}$
Elliptical	$\begin{pmatrix} \cos\chi \\ \sin\chi \, \mathrm{e}^{j\phi} \end{pmatrix}$	$\begin{pmatrix} 1 \\ \cos 2\varepsilon \cos 2\alpha \\ \cos 2\varepsilon \cos 2\alpha \\ \sin 2\varepsilon \end{pmatrix}$	$\begin{pmatrix} c^2 & \mathrm{e}^{-j\phi} sc \\ \mathrm{e}^{j\phi} sc & s^2 \end{pmatrix}$ $c = \cos\chi$ $s = \sin\chi$
Unpolarized	none	$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$\frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

All vectors are normalized to a Jones vector of unit length.

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