Fundamental Principles of Lasers and Ultrafast Optics

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Chapter 1

Photons and Radiation

1.1 Wave and Mode

In classical mechanics, wave is a simple form of collective motion whose closest analogy is vibration. Consider a one-dimensional string of bounded particles in which each particle's motion is influenced by its two nearest neighbors. If the motion of the particles around its equilibrium point is small, one can assume the force acting on each particle is proportional to the separation between the particles. The equation of motion reads:

$$m\ddot{a}_l = \eta(a_{l-1} - a_l) + \eta(a_{l+1} - a_l), \tag{1.1}$$

where m is the mass of the particles, a_l the displacement of the lth particle from the equilibrium point, and η the spring constant. This system of linear equations can be solved by a change of variable (Fourier transformation)

$$a_l(t) = \sum_k u_k(t)\exp(ikls), \tag{1.2}$$

where s is the distance between neighboring particles. The new equations are:

$$\sum_k m\ddot{u}_k e^{ilks} = 2\eta\sum_k[\cos(ks) - 1]u_k e^{ilks}. \tag{1.3}$$

Comparing coefficients, we have

$$m\ddot{u}_k = 2\eta[\cos(ks) - 1]u_k, \tag{1.4}$$

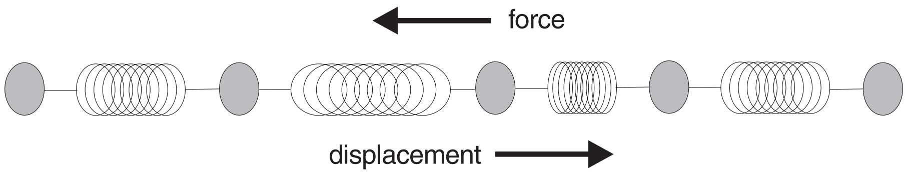

Figure 1.1: A model of mechanical waves.

hence

$$u_k(t) = u_k(0)\cos(\omega t) + \dot{u_k}(0)\sin(\omega t), \tag{1.5}$$

where $m\omega^2 = 2\eta[1 - \cos(ks)]$ is called the dispersion relation of the wave. Under the new variables u_k, the equations of motion are decoupled. Each degree of freedom u_k is an independent harmonic oscillator. Each such collective harmonic motion, labeled by k, is called a mode. In fact, for each k there are three degrees of freedom, because the motion of the particles has two directions perpendicular to the string and one parallel to the string. In a continuous system, $s \to 0$, the dispersion relation becomes $\omega = k\sqrt{\eta s^2/m}$. The phase velocity ω/k is independent of ω. In the phasor notation,

$$a_l(t) = \sum_k \tilde{u}_k(0)e^{i(kx-\omega t)}, \tag{1.6}$$

where $x = ls$ is the spatial coordinate, and $\tilde{u}_k(0)$ is the initial complex amplitude. It is seen that the displacement of the lth particle is a superposition of various modes (waves).

1.2 Boltzmann Distribution

In a system of many degrees of freedom, nonlinear coupling can cause energy transfer between modes. Even if the 10^{23} coupled equations of motion can be solved by some supercomputer in the future, there will be too much information, useful or not, in the solutions to be digested by a human being. In most cases, we are not so concerned with the individual motion of each degree of freedom. Instead, we are concerned with the average dynamics. The key point is that each macroscopic state of matter corresponds to an extremely large number of microscopic states. Some macroscopic states are more probable than others because they correspond to a larger number of microscopic states. In order to find an economic way to connect macroscopic states with microscopic states, Boltzmann postulated that for an interactive system at equilibrium, the system has an equal probability in every microscopic state. This revolutionary idea led to a powerful method—statistical mechanics.

Let us consider a macroscopic system made of a large number of microscopic parts, each in a different microscopic state. For simplicity, let us label each microscopic part only by its energy E_i. Assume there are n_i microscopic systems whose energy is E_i. One has

$$\sum_i n_i = N, \tag{1.7}$$

$$\sum_i n_i E_i = E, \tag{1.8}$$

where N is the number of microscopic systems and E is the total energy. For each set $\{n_i\}$ there are $W(\{n_i\})$ ways of constructing the macroscopic system.

$$W(\{n_i\}) = \frac{N!}{\prod_i n_i!} \tag{1.9}$$

According to the postulate of equal *a priori* probability, a set $\{n_k\}$ is less probable than another set $\{n_l\}$ if $W(\{n_k\})$ is smaller than $W(\{n_l\})$. Therefore the most probable distribution of $\{n_i\}$ is the one that maximize W under the constraint of Eqs. (1.7) and (1.8). By the method of Lagrange multipliers

$$\frac{\partial}{\partial n_i}\left[\ln W(\{n_i\}) - \alpha \sum_i n_i - \beta \sum_i n_i E_i\right] = 0. \tag{1.10}$$

Using Stirling's approximation $\ln N! \approx N \ln N - N$,

$$\ln W(\{n_i\}) \approx N \ln N - \sum_i n_i \ln n_i. \tag{1.11}$$

Eq. (1.10) becomes

$$-\ln n_i - \alpha - \beta E_i = 0, \tag{1.12}$$

hence $n_i \propto \exp(-\beta E_i)$, which means the probability of finding a microscopic system with energy E_i is proportional to $\exp(-\beta E_i)$. As we shall see, $\beta = 1/(kT)$. This is known as the Boltzmann distribution.

Now consider two macroscopic systems, the total number of ways of constructing the two systems is $W(\{n_i\}) = W^{(1)}(\{n_i^{(1)}\})W^{(2)}(\{n_i^{(2)}\})$. If they are isolated, the individual distributions are determined by

$$\frac{\partial}{\partial n_i^{(1)}}\left[\ln W^{(1)}(\{n_i^{(1)}\}) - \alpha^{(1)} \sum_i n_i^{(1)} - \beta^{(1)} \sum_i n_i^{(1)} E_i^{(1)}\right] = 0 \tag{1.13}$$

and

$$\frac{\partial}{\partial n_i^{(2)}}\left[\ln W^{(2)}(\{n_i^{(2)}\}) - \alpha^{(2)}\sum_i n_i^{(2)} - \beta^{(2)}\sum_i n_i^{(2)}E_i^{(2)}\right] = 0 \tag{1.14}$$

respectively. If the two systems are brought to thermal equilibrium, there is only one constraint of total energy, $\sum_i n_i^{(1)}E_i^{(1)} + \sum_i n_i^{(2)}E_i^{(2)} = E$, instead of two independent energy constraints $\sum_i n_i^{(1)}E_i^{(1)} = E^{(1)}, \sum_i n_i^{(2)}E_i^{(2)} = E^{(2)}$. Fewer constraint means more possibility. Therefore $W_{\text{eq}} \geq W_{\text{iso}}$. Equality occurs only if $\beta^{(1)} = \beta^{(2)}$ initially. This means when a system approaches equilibrium, $\ln W$ increases until the β of all its parts becomes equal. This is a strong indication that $\ln W$ is related to entropy and β is a measure of temperature.

Now consider a microscopic system in thermal contact with a macroscopic system with fixed β. Let the energy of the microscopic system be E_s, and the energy of the macroscopic system be E_r. From Boltzmann distribution, the probability of $E_s = E_0 - dE$ over the probability of $E_s = E_0$ is $\exp(\beta dE)$. This means for the macroscopic system, $W(E_r + dE)/W(E_r) = \exp(\beta dE)$, *i.e.*,

$$d\ln W \equiv \ln W(E_r + dE) - \ln W(E_r) = \beta dE. \tag{1.15}$$

Here the small amount of energy dE can be viewed as the thermal energy transferred from the macroscopic system to the microscopic system. In thermodynamics this is described by the following equation:

$$dS = \frac{1}{T}dE, \tag{1.16}$$

where S is the entropy. Comparing with Eq. (1.15), one has $\ln W = S/k$ and $\beta = 1/(kT)$, where k is the Boltzmann constant.

By calculating W as a function of macroscopic variables, such as the energy E, the volume V etc., one can obtain the entropy function $S(E,V)$. From $S(E,V)$ one can obtain the equation of macroscopic state. This is how statistical mechanics works. Let us calculate the equation of state for the ideal gas as an example. Envision N independent particles in a box of volume V. For each particle the number of possible states is proportional to V. Because the particles move independently, the number of microscopic states for the whole system is proportional to V^N. In addition, the particles can have different momenta subjected to the constraint $\sum_{i=1}^{3N} p_i^2/(2m) = E$. Considering the surface area of a $3N$-dimensional sphere, the number of states with energy $= E$ is proportional to $\sqrt{E}^{3N-1}$. Therefore one has $W(E,V) =$

$CV^N\sqrt{E}^{3N-1}$, where C is a proportional constant which depends only on N and m. $S(E,V) = k\ln W(E,V) \approx kN\ln V + 3kN\ln E/2 + k\ln C(N)$. From the thermodynamic relation $dE = TdS - PdV$ one has $dS = (1/T)dE + (P/T)dV$, hence $P/T = \partial S/\partial V = kN/V$, $1/T = \partial S/\partial E = 3kN/(2E)$, *i.e.*, $PV = NkT$, $E = 3NkT/2$.

The entropy function is not always the most convenient to start with. For example, in studying a system which is in thermal contact with a large heat bath, it is more convenient to describe the system as a function of (T,V) instead of (E,V) because T is fixed by the temperature of the bath. Therefore one uses the Helmholtz free energy $A = E - TS$, in view of $dA = -SdT - PdV$. Since the system can exchange energy with the heat bath, its energy is not a fixed value as in an isolated system. Instead the energy follows a probability distribution, namely the Boltzmann distribution. The expectation value is

$$\bar{E} = \frac{\sum_{E_i} E_i W(N,V,E_i)e^{-E_i/kT}}{\sum_{E_i} W(N,V,E_i)e^{-E_i/kT}}. \tag{1.17}$$

Let us define the canonical partition function

$$Q(N,V,T) \equiv \sum_{E_i} W(N,V,E_i)e^{-E_i/kT}, \tag{1.18}$$

one has

$$\bar{E} = \frac{-\partial \ln Q}{\partial \beta}. \tag{1.19}$$

On the other hand,

$$E = A - T\frac{\partial A}{\partial T} = -T^2\frac{\partial (A/T)}{\partial T} = \frac{\partial (A/kT)}{\partial \beta}, \tag{1.20}$$

therefore $A = -kT\ln Q$. By calculating Q from the microscopic states, one obtains the macroscopic quantity A.

1.3 Rayleigh-Jeans Formula for Blackbody Radiation

How matter radiates light has been an intriguing problem in physics for a long time. In the beginning of the 20th century, Rayleigh and Jeans attempted to understand the spectrum of blackbody radiation from classical statistical mechanics. Treating light as classical electromagnetic waves, they first calculated the mode density of light waves, then assigned each mode an average energy kT as one would do for mechanical waves. This way they obtained a formula for the power spectrum of blackbody radiation.

Assume light waves exist only in an isolated space with length, width and height all equal to L. Then only light waves which satisfy the following boundary condition exist.

$$k_i L = \quad n_i \pi \quad i = x, y, z. \tag{1.21}$$

where k_i are the components of the wave vector and n_i are positive intergers. The condition for light waves with frequency smaller than ν_m is

$$\frac{c}{2\pi}\sqrt{k_x^2 + k_y^2 + k_z^2} \leq \nu_m. \tag{1.22}$$

or

$$\sqrt{n_x^2 + n_y^2 + n_z^2} \leq \frac{2L\nu_m}{c}. \tag{1.23}$$

The total number of modes with frequency lying between 0 and ν_m is

$$2 \times \frac{1}{8}\frac{4\pi}{3}\left(\frac{2L\nu_m}{c}\right)^3 = \frac{8\pi}{3}\frac{L^3\nu_m^3}{c^3}. \tag{1.24}$$

The factor 2 in the above expression accounts for the two polarization of light. The number of modes per unit volume in the frequency interval between ν and $\nu + d\nu$ is therefore

$$\frac{1}{L^3}\frac{d}{d\nu}\left(\frac{8\pi}{3}\frac{L^3\nu_m^3}{c^3}\right) d\nu = \frac{8\pi\nu^2}{c^3} d\nu. \tag{1.25}$$

From Boltzmann distribution the average energy per mode is

$$\frac{\int\int \left(\frac{p^2}{2m} + \frac{kq^2}{2}\right) \exp\left[-\frac{1}{kT}\left(\frac{p^2}{2m} + \frac{kq^2}{2}\right)\right] dpdq}{\int\int \exp\left[-\frac{1}{kT}\left(\frac{p^2}{2m} + \frac{kq^2}{2}\right)\right] dpdq} = kT. \tag{1.26}$$

Multiplying Eqs. (1.25) and (1.26), the power spectrum of the light is

$$P(\nu)d\nu = \frac{8\pi\nu^2 kT}{c^3} d\nu. \tag{1.27}$$

One can immediately recognize a serious problem in Eq. (1.27). If the equation were correct, then any matter at any finite temperature would emit much more X-ray than visible light, not to mention the γ-rays! The power spectrum diverges at high frequency. Needless to say, such a spectrum does not agree with experimental observations. But Eq. (1.27) is derived from very fumdamental principles, it was hard to understand what went wrong. It turned out that the quantum nature of light has a subtle but important effect on the blackbody spectrum. The problem was solved by Max Planck with his revolutional hypothesis of quantized light energy.

1.4 Planck's Theory of Blackbody Radiation

Planck showed that if light energy is not continuously variable, instead, if the energy level spacing for light with frequency ν is

$$\Delta E = h\nu, \tag{1.28}$$

then a blackbody spectrum which is free of divergence and agrees with experimental observations can be derived using the same idea of statistical mechanics. Using Eq. (1.28), the average energy per mode becomes

$$\frac{\sum_{n=0}^{\infty} nh\nu \exp\left[-nh\nu/(kT)\right]}{\sum_{n=0}^{\infty} \exp\left[-nh\nu/(kT)\right]} = \frac{h\nu}{\exp\left[h\nu/(kT)\right] - 1}. \tag{1.29}$$

The spectrum of light is then

$$P(\nu)d\nu = \frac{8\pi\nu^2}{c^3} \frac{h\nu}{\exp\left[h\nu/(kT)\right] - 1} d\nu. \tag{1.30}$$

The rapid decrease of average mode energy with increasing frequency saves the spectrum from diverging at high frequency.

1.5 Einstein A and B Coefficients

With the development of the idea of quantized photon energy and quantized energy states in atoms and molecules, it became apparent that photons can induce state transitions on atoms and molecules. Reversely, when atoms and molecules make spontaneous state transitions, they often emit photons. Thus any fundamental theory of radiation must describe the roles of atoms and molecules in the processes of absorption and emission of photons. In 1917, Einstein proposed that the transition rate, $R_{2\to1}(\mathbf{k})$, from the upper state to the lower state by emitting a photon to mode $\mathbf{k}$ is equal to the spontaneous emitting rate A_{21} plus the stimulated emitting rate $B_{21}N(\mathbf{k})$ from the upper state to the lower state, *i.e.*,

$$R_{2\to 1}(\mathbf{k}) = A_{21} + B_{21}N(\mathbf{k}), \tag{1.31}$$

and the transition rate from the lower state to the upper state, $R_{1\to 2}(\mathbf{k})$, by absorbing a photon from mode $\mathbf{k}$ is equal to the stimulated absorbing rate $B_{12}N(\mathbf{k})$ from the lower state to the upper state, *i.e.*,

$$R_{1\to 2}(\mathbf{k}) = B_{12}N(\mathbf{k}), \tag{1.32}$$

where $N(\mathbf{k})$ is the photon number of mode $\mathbf{k}$, and A, B are constants to be determined. Let the population of the upper state be N_2, and that of the lower state be N_1, and assume that the number of photons in modes other than $\mathbf{k}$ is negligible. If the atomic or molecular system is in a steady state, then from the principle of detailed balancing we have

$$N_1 R_{1\to 2} = N_2 R_{2\to 1}. \tag{1.33}$$

Solving Eq. (1.33) for $N(\mathbf{k})$, we obtain

$$N(\mathbf{k}) = \frac{A_{21}/B_{21}}{(N_1 B_{12})/(N_2 B_{21}) - 1}. \tag{1.34}$$

At thermal equilibrium

$$\frac{N_1}{N_2} = \exp\left(\frac{E_2 - E_1}{kT}\right) = \exp\left(\frac{h\nu}{kT}\right), \tag{1.35}$$

Eq. (1.34) becomes

$$N(\mathbf{k}) = \frac{A_{21}/B_{21}}{(B_{12}/B_{21})\exp\left[h\nu/(kT)\right] - 1}. \tag{1.36}$$

From Planck's theory of blackbody radiation, the photon number for a *specific photon mode* $\mathbf{k}$ at thermal equilibrium is

$$N(\mathbf{k}) = \frac{1}{\exp\left[h\nu/(kT)\right] - 1}. \tag{1.37}$$

Therefore, if the matter and light are at the same thermal equilibrum temperature, by comparing Eqs. (1.36) and (1.37) we have

$$\frac{A_{21}}{B_{21}} = \frac{B_{12}}{B_{21}} = 1. \tag{1.38}$$

This is how Einstein derived the relation between spontaneous emission and stimulated emission without using quantum mechanics. The element of quantum mechanics is hidden in the Planck's theory of blackbody radiation, namely $\Delta E = h\nu$.

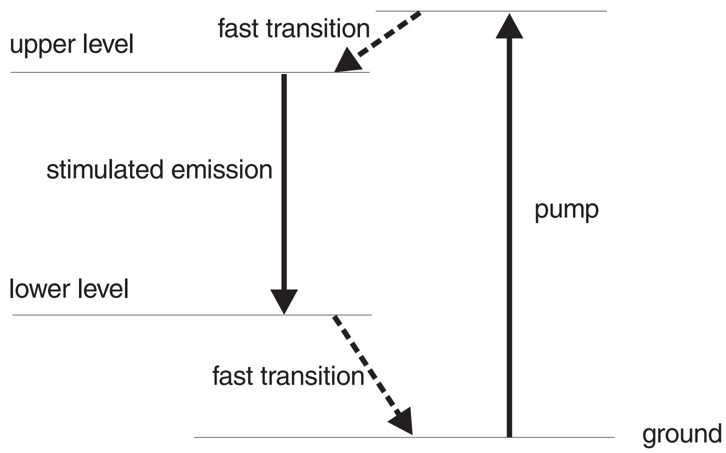

Figure 1.2: Energy diagram of 4-level lasers.

1.6 Laser Rate Equations

Assume the number of photons in modes other than $\mathbf{k}$ are negligible, from the results of the last section we may write down the following rate equations.

$$\frac{dN_2}{dt} = -B(N_2 - N_1)N(\mathbf{k}) - AN_2, \tag{1.39}$$

$$\frac{dN_1}{dt} = B(N_2 - N_1)N(\mathbf{k}) - A'N_1, \tag{1.40}$$

$$\frac{dN(\mathbf{k})}{dt} = B(N_2 - N_1)N(\mathbf{k}) + BN_2, . \tag{1.41}$$

In the above equations, the first terms in the right-hand side represent the stimulated process, and the second terms the spontaneous process. A is much larger than B, because when the population of the excited state is decreased by spontaneous emission, photons can be emittd into many modes other than mode $\mathbf{k}$. Note that $A' \neq 0$ unless the lower level is the ground state. At thermal equilibrium $N_1 > N_2$, therefore in general conditions matter absorbs light. However, light can be amplified if one can maintain the nonequilibrium condition $N_2 > N_1$. In order to do that, an external pump is required to bring the population in the lower state to the upper state constantly. It is much easier to maintain a constant population inversion condition $N_2 > N_1$ if the lower level is not the ground state and has a much shorter lifetime than the upper level. Assume the decrease of N_1 due to spontaneous emission is much faster than its increase due to stimulated emission from the upper level, so that N_1 is always close to 0, then we may omit Eq. (1.40) and rewrite Eq. (1.39) and Eq. (1.41) by adding an external pump into the following form.

$$\frac{dN_2}{dt} = -BN_2N - \frac{N_2}{T_1} + \Gamma, \tag{1.42}$$

$$\frac{dN}{dt} = BN_2(N+1), \tag{1.43}$$

where Γ represents the contribution from pumping, and T_1 is the upper-state lifetime. Let us assume the cross section and the length of the light beam is A and L respectively. We may elucidate the physical meaning of Eqs. (1.42) and (1.43) by making the following change of variables: $n = N/AL$ is the photon density, $\Delta N = N_2/AL$ is the population inversion density, and $\gamma = \Gamma/AL$ is the pump density. Under the lasing condition $N \gg 1$, Eqs. (1.42) and (1.43) become

$$\frac{d\Delta N}{dt} = -BAL\Delta Nn - \frac{\Delta N}{T_1} + \gamma, \tag{1.44}$$

$$\frac{dn}{dt} = BAL\Delta Nn. \tag{1.45}$$

Because B has the dimension 1/sec, we may recast Eqs. (1.44) and (1.45) into the following form.

$$\frac{d\Delta N}{dt} = -c\sigma\Delta Nn - \frac{\Delta N}{T_1} + \gamma, \tag{1.46}$$

$$\frac{dn}{dt} = c\sigma\Delta Nn, \tag{1.47}$$

where c is the speed of light. Note that nc is the photon flux, and σ is the emission/absorption cross section. The physical meaning of Eqs. (1.46) and (1.47) thus become apparent. The parameters σ and T_1 have simpler physical meaning than the A and B coefficients, and can be readily measured with experiments.

Eqs. (1.46) and (1.47) describe the dynamics of light amplification and population inversion in a laser medium. Because amplified light can quickly propagate out of the physical region of the gain medium, practical lasers are usually constructed with an optical cavity to confine the light. To make a laser useful, one must purposely couple out part of the light from the cavity. The coupling introduces photon loss in the laser. Loss of photon may also be due to inperfections of the laser cavity, such as diffraction from the edges of mirrors or absorption by impurities. To take into account the photon loss in the laser, we add a phenomenological photon-loss term to Eq. (1.47). The term is characterized by a time constant T_c, the cavity lifetime.

$$\frac{dn}{dt} = c\sigma\Delta N n - \frac{n}{T_\mathrm{c}}. \tag{1.48}$$

For a laser with round-trip length L and round-trip loss l,

$$\frac{1}{T_\mathrm{c}} = \frac{-\ln(1-l)c}{L}. \tag{1.49}$$

Eqs. (1.46) and (1.48) are called the laser rate equations.

Exercise 1.1. Calculate the ratio A/B in Eq. (1.39) by considering the density of states in the spontaneous emission process.

1.7 Threshold Pumping Rate and Slope Efficiency

From the laser rate equations we may derive the steady state operating conditions of a laser. There are two steady state solutions of Eqs. (1.46) and (1.48). One is

$$\Delta N = \gamma T_1, \tag{1.50}$$

$$n = 0. \tag{1.51}$$

The other is

$$\Delta N = \frac{1}{c\sigma T_\mathrm{c}}, \tag{1.52}$$

$$n = \left(\gamma - \frac{1}{c\sigma T_1 T_\mathrm{c}}\right) T_\mathrm{c}. \tag{1.53}$$

When $\gamma > 1/(c\sigma T_1 T_\mathrm{c})$, $n > 0$. The output intensity is

$$I_\mathrm{out} = \left(\gamma - \frac{1}{c\sigma T_1 T_\mathrm{c}}\right) T_\mathrm{c}(1-R)ch\nu, \tag{1.54}$$

where R is the reflectivity of the output coupler mirror. Eq. (1.54) shows that after the pumping rate reaches a threshold value $1/c\sigma T_1 T_\mathrm{c}$ the output of the laser increases linearly with the pumping rate. The slope efficiency, defined as $dP_\mathrm{out}/dP_\mathrm{in}$, where P_out is the output power and P_in the pump power, is proportional to $T_\mathrm{c}(1-R)ch\nu$. Note that unlike the threshold pumping rate, the slope efficiency is independent of σ and T_1.

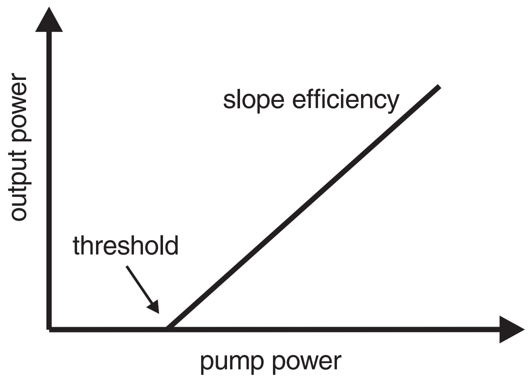

Figure 1.3: Laser output as a function of pump power.

We may estimate the threshold pumping rate required for lasing in a Nd:YAG laser crystal. Assume the length of the crystal is 6 cm, and the end reflectivity is 6%. Then the photon loss rate is $-3 \times 10^8 \times \ln(0.06 \times 0.06)/0.12 = 1.4 \times 10^{10}$ s^{-1}. $T_1 = 230$ μs and $\sigma = 6.5 \times 10^{-19}$ cm^2 for the Nd atoms in the YAG host crystal. We have

$$\gamma_{\text{th}} = \frac{1.4 \times 10^{10}}{3 \times 10^8 \times 6.5 \times 10^{-23} \times 230 \times 10^{-6}} = 3.1 \times 10^{27} \text{ /sec} \cdot \text{m}^3.$$

Assume the crystal diameter is 6 mm, the pumping efficiency (power absorbed by the crystal/pump power) is 8%, and the mean absorption wavelength λ is 500 nm, then the total pump power required is as large as

$$\frac{\gamma_{\text{th}} \times \pi \times (0.003)^2 \times 0.06 \times hc/\lambda}{0.08} = 2.6 \times 10^4 \text{ W}.$$

Therefore, in practice photons must be confined in an optical resonant cavity to reduce the photon loss rate before lasing can happen. We shall see in the following chapters that it is also the resonant cavity which gives a laser its unique optical properties—temporal and spatial coherence.

We have derived the laser rate equations using Einstein's radiation theory. In chapter 4 we shall derive the rate equations directly from the Shrödinger equation and the Maxwell equations.

Exercise 1.2. Using the parameters of Nd:YAG laser given above, design a Nd:YAG laser which gives maximum output power. The cavity length of the laser is 1 m, the pump power is 2 kW, and the internal loss is 5%. Determine the reflectivity of the output coupler mirror which yields maximum power.

1.8 Gain Saturation

The gain of a laser amplifier of length L is defined by

$$I_{\text{out}} = I_{\text{in}} \exp(gL). \tag{1.55}$$

From Eq. (1.47),

$$g = \sigma \Delta N. \tag{1.56}$$

From Eq. (1.46),

$$g = \frac{\sigma\gamma}{c\sigma n + 1/T_1}. \tag{1.57}$$

Eq. (1.57) can be written in the following form:

$$g = \frac{g_0}{1 + I/I_s}, \tag{1.58}$$

where g_0 is called the small signal gain and I_s the saturation intensity.

$$g_0 = \sigma T_1 \gamma, \tag{1.59}$$

$$I_s = \frac{h\nu}{T_1 \sigma}. \tag{1.60}$$

Note that there is a trade-off between small signal gain and saturation intensity. The larger the emission cross section, the larger the gain, and unavoidably, the smaller the saturation intensity. Laser media with large $T_1\sigma$ have low threshold pumping rates, but the amplification also saturates at low intensity.

1.9 Starting from Spontaneous Emission

Lasers are self-starting devices. Spontaneous emission provides the initial photons which are multiplied subsequently by stimulated emission. To take into account of spontaneous emission, we rewrite Eq. (1.48) in a more accurate form according to Eq. (1.43).

$$\frac{dn}{dt} = c\sigma\Delta N \left(n + \frac{1}{V}\right) - \frac{n}{T_{\text{c}}}. \tag{1.61}$$

where V is the volume of the cavity. Since in the starting stage the photon density is small, the population inversion density ΔN is approximately constant. Assume the solution to Eq. (1.61) has the form $n(t) = \tilde{n}(t)\exp(\alpha t)$,

$\alpha \equiv c\sigma\Delta N - 1/T_{\mathrm{c}}$, we obtain

$$\tilde{n}(t) = \frac{c\sigma\Delta N}{V\alpha}\left[1 - \exp(-\alpha t)\right]. \tag{1.62}$$

Hence

$$n(t) = \frac{c\sigma\Delta N}{V\alpha}\left[\exp(\alpha t) - 1\right]. \tag{1.63}$$

In the limit $t \to 0$, $n(t) \approx (c\sigma\Delta N/V)t$.

Chapter 2

Quantum Theory of Radiation

2.1 Wave Mechanics

In 1911 Rutherford showed by α particle scattering that most of the positive charge in an atom is concentrated in the center, called the nucleus. But such an atomic model cannot explain the stability of atoms. According to classical mechanics, the only way for the electrons not to fall into the nucleus is by circulating. However, according to the classical electromagnetic theory, circulating electrons lose energy by emitting a continuous spectrum of electromagnetic wave, therefore cannot maintain their orbits. Even more puzzling is the experimental fact that atomic spectra consist of sharp discrete ordered lines that no theory was able to explain. In 1913 Bohr discovered a rule for the quantization of electron orbits,

$$L = n\hbar, \tag{2.1}$$

where L is the angular momentum n is a positive integer. This simple rule accurately describes the hydrogen spectrum. But why the electron orbits follow this rule was still not explained. In 1916 Wilson and Sommerfeld found a more general quantization condition,

$$\oint p(q)\, dq = nh, \tag{2.2}$$

where p is the momentum and q is the coordinate. The Wilson-Sommerfeld quantization rule not only unifies Planck's quantization rule for the harmonic oscillator and Bohr's for the hydrogen atom, it can be applied to any integrable system (where p can be solved as a function of q). An important achievement of the Wilson-Sommerfeld quantization rule is that when combined with relativity, it explained accurately the fine structure of the hydrogen spectrum (the dependence of energy levels on the angular momentum).

It was not until 1924, when de Broglie presented the idea of matter wave, started the wave mechanics. de Broglie noted that if an electron moves like a wave, with wavelength $\lambda = h/p$, then the Wilson-Sommerfeld quantization rule is simply the common periodic condition for waves, and the photon quantization rule $E = h\nu$ is just a special case. This astonishing idea was soon verified by electron diffraction experiments in 1927 (Davisson and Germer, G. P. Thomson). Finally, one saw a good reason behind the Wilson-Sommerfeld quantization rule. In 1925 Schrödinger derived the matter wave equation by replacing p with $-i\hbar\partial/\partial q$ and E with $i\hbar\partial/\partial t$ in the classical relation $p^2/(2m) + V(q) = E$. The result is the Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial q^2} + V(q)\psi = i\hbar\frac{\partial\psi}{\partial t}. \tag{2.3}$$

The replacement was based on the obvious fact that when $-i\hbar\partial/\partial q$ operates on a wave of wave number k, it multiplies the wave by $\hbar k = p$, and similarly for the energy operator $i\hbar\partial/\partial t$. As a matter of fact, the first quantum wave equation Schrödinger derived was the relativistic wave equation, now known as the Klein-Gordon equation,

$$-\hbar^2\frac{\partial^2\psi}{\partial t^2} = -\hbar^2 c^2\nabla^2\psi + m^2c^4\psi, \tag{2.4}$$

which was based on the relativistic energy-momentum relation $E^2 = \mathbf{p}^2c^2 + m^2c^4$. However, when he added the electromagnetic potential $(\phi, \mathbf{A}/c)$, Eq. (2.4) did not give the correct fine structure of the hydrogen spectrum. This forced Schrödinger to fall back to the non-relativistic approximation Eq. (2.3). (Now we know the reason Eq. (2.4) does not work is because the electron has spin, therefore its wavefunction cannot be a scalar.)

Consider a general normalized matter wave $u(x,t)$ which is a linear combination of monochromatic waves $\exp(ikx)$. Each of the component waves has a definite wavenumber k.

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int \tilde{u}(k,t)e^{ikx}\,dk, \tag{2.5}$$

where $\tilde{u}(k,t) = a(k)\exp[-i\omega(k)t]$ and

$$\int u^*(x,t)u(x,t)\,dx = \int \tilde{u}^*(k,t)\tilde{u}(k,t)dk = 1. \tag{2.6}$$

Because the relative strength of the wave component $\exp(ikx)$ is represented by $|\tilde{u}(k,t)|^2$, the expectation value of $p = \hbar k$ is simply

$$\hbar\bar{k} = \int \hbar k \tilde{u}^*(k,t)\tilde{u}(k,t)\,dk = \int u^*(x,t)\hat{p}u(x,t)\,dx, \tag{2.7}$$

where the momentum operator $\hat{p}$ is equal to $-i\hbar\partial/\partial x$, and the second equality is derived from

$$\begin{aligned}
&\int u^*(x,t)\left[-i\hbar\frac{\partial}{\partial x}u(x,t)\right]\,dx \\
=\;& \frac{1}{2\pi}\int\left(\int \tilde{u}^*(k,t)e^{-ikx}\,dk\right)\left(\int \hbar k'\tilde{u}(k',t)e^{ik'x}\,dk'\right)dx \\
=\;& \int\int \tilde{u}^*(k,t)\delta(k-k')\hbar k'\tilde{u}(k',t)\,dkdk' \\
=\;& \int \hbar k\tilde{u}^*(k,t)\tilde{u}(k,t)\,dk.
\end{aligned} \tag{2.8}$$

By the symmetric roles of x and k in Fourier analysis, the expectation value of x is

$$\begin{aligned}
\bar{x} &= \int \tilde{u}^*(k,t)\frac{i\partial}{\partial k}\tilde{u}(k,t)\,dk \\
&= \frac{1}{2\pi}\int\left(\int u^*(x,t)e^{ikx}\,dx\right)\left(\int x'u(x',t)e^{-ikx'}\,dx'\right)dk \\
&= \int\int u^*(x,t)\delta(x-x')x'u(x',t)\,dxdx' \\
&= \int u^*(x,t)xu(x,t)\,dx
\end{aligned} \tag{2.9}$$

If one agrees that the probability distribution function of k is $\tilde{u}^*(k)\tilde{u}(k)$, then the probability distribution function of x is $u^*(x,t)u(x,t)$. Eqs. (2.7) and (2.9) point out a general rule: the expectation value of a physical quantity $O(x,p)$ is simply $\bar{O} = \int u^*(x,t)\hat{O}u(x,t)\,dx$, where $\hat{O} = O(x, -i\hbar\partial/\partial x)$. If one expands $u(x,t)$ with respect to an orthonormal basis set made of eigenfunctions of $\hat{O}$, *i.e.* $u(x,t) = \sum_i a_i(t)s_i(x)$ and $\hat{O}s_i(x) = O_i s_i(x)$, then

$$\bar{O} = \sum_i O_i|a_i|^2. \tag{2.10}$$

Eq. (2.10) points out that the physical quantity O can have values $O_1, O_2, O_3, \ldots$ with probability $|a_1|^2, |a_2|^2, |a_3|^2, \ldots$ respectively. If a basis $v_i(x)$ which is not a set of $\hat{O}$'s eigenfunctions is used, $u(x,t) = \sum_i b_i(t)v_i(x)$, the expectation value becomes

$$\begin{aligned} \bar{O} &= \sum_{i,j} b_i^* b_j \int v_i^*(x)\hat{O}v_j(x)\,dx \\ &\equiv \sum_{i,j} b_i^* O_{ij} b_j, \end{aligned} \tag{2.11}$$

where O_{ij} is a matrix representation of the operator $\hat{O}$ with respect to the basis $v_i(x)$.

Now it should be clear that in wave mechanics the state of a physical system is represented by the wavefunction $u(x,t)$, which is a solution of the Schrödinger equation, and the value of a physical quantity is represented by its spectrum O_i and probability distribution $|a_i|^2$. The wavefunction can have different forms (such as $u(x,t), \tilde{u}(k), a_i(t)$), depending on which basis is used, so can physical quantities ($\hat{p} = \hbar k$ or $-i\hbar\partial/\partial x$). Because in most cases of formal discussion it is not necessary to specify the basis, one can simply represent a wavefunction by $|s\rangle$, its complex conjugate by $\langle s|$, the expectation value of an operator by $\langle s|\hat{O}|s\rangle$, and the matrix element linking $|s'\rangle$ and $|s\rangle$ by $\langle s'|\hat{O}|s\rangle$. This is called the Dirac notation.

2.2 Canonical Quantization

A limitation of Schrödinger equation is that the energy of the system must be of the form $p^2/(2m) + V(q)$. This is not the case for many systems described by generalized momenta and coordinates. In fact, the form of the energy function can be changed by canonical transformations, and it is known that systems linked by canonical transformations are equivalent in classical mechanics. How should one quantize a system with an arbitrary Hamiltonian (energy) function? Dirac postulated that classical systems can be quantized by the following procedure, which is called canonical quantization:

1. Construct the classical Hamiltonian function H, and find out the canonical coordinates q_i and the canonical momenta p_i. In these variables the equation of motion is

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}. \tag{2.12}$$

2. Interpret the canonical coordinates q_i, and momenta p_j as operators and impose the commutation relations

$$[q_i, p_j] \equiv q_i p_j - p_j q_i = i\hbar\delta_{ij} I. \tag{2.13}$$

3. Considering the operators being static (the Shrödinger representation), the equation of motion for the states is

$$i\hbar\frac{d}{dt}|t\rangle = H|t\rangle. \tag{2.14}$$

   Or alternatively, considering the states being static (the Heisenberg representation), the equation of motion for the operators is

$$i\hbar\frac{dO(t)}{dt} = [O, H] + i\hbar\frac{\partial O}{\partial t}. \tag{2.15}$$

Eq. (2.15) can be obtained from Eq. (2.14).

$$\begin{aligned} i\hbar\frac{d}{dt}\langle t|O|t\rangle &= \left(i\hbar\frac{d}{dt}\langle t|\right) O|t\rangle + \langle t|\left(i\hbar\frac{\partial}{\partial t}O\right)|t\rangle + \langle t|O\left(i\hbar\frac{d}{dt}|t\rangle\right) \\ &= \langle t|\left([O, H] + i\hbar\frac{\partial O}{\partial t}\right)|t\rangle. \end{aligned} \tag{2.16}$$

2.3 Simple Harmonic Oscillator

The Hamiltonian function of a simple harmonic oscillator is

$$H = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2}. \tag{2.17}$$

Define the annihilation operator a and the creation operator $a^\dagger$ by

$$a \equiv \sqrt{\frac{m\omega}{2\hbar}}\left(q + i\frac{p}{m\omega}\right), \qquad a^\dagger \equiv \sqrt{\frac{m\omega}{2\hbar}}\left(q - i\frac{p}{m\omega}\right), \tag{2.18}$$

then

$$aa^\dagger - a^\dagger a = 1, \tag{2.19}$$

and

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right). \tag{2.20}$$

Let us label the eigenvalues of the operator $a^\dagger a$ by λ_n, and the corresponding eigenvector by $|n\rangle$, that is

$$a^\dagger a|n\rangle = \lambda_n|n\rangle. \tag{2.21}$$

Since $\langle n|a^\dagger a|n\rangle$ is the norm of $a|n\rangle$, it follows $\lambda_n \geq 0$. If $|n\rangle$ is an eigenvector of $a^\dagger a$, then $a|n\rangle$ and $a^\dagger|n\rangle$ are also eigenvectors, as can be seen from the following equations.

$$\left(a^\dagger a\right) a|n\rangle = \left(aa^\dagger - 1\right) a|n\rangle = (\lambda_n - 1)\, a|n\rangle, \tag{2.22}$$

and similarly,

$$\left(a^\dagger a\right) a^\dagger|n\rangle = (\lambda_n + 1)\, a^\dagger|n\rangle. \tag{2.23}$$

By applying a successively, new eigenvectors with smaller and smaller eigenvalues can be derived. However, since the eigenvalues can not be smaller than zero, this process must stop at some eigenvector which has the lowest possible eigenvalue. This particular eigenvector is called the ground state $|0\rangle$. Since there can be no eigenvector with smaller eigenvalue, $a|0\rangle$ must not be another eigenvector. The only choice is then $a|0\rangle = 0$ and thus $\lambda_0 = 0$. It follows from Eq. (2.23) that $\lambda_n = n$, and the eigenvalues of the Hamiltonian are $E_n = \hbar\omega\,(n + 1/2)$.

In the Schrödinger representation,

$$i\hbar\frac{d}{dt}|n\rangle = H|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle. \tag{2.24}$$

The solution is

$$|n\rangle = e^{-i\omega(n+1/2)}|n\rangle_0, \tag{2.25}$$

where $|n\rangle_0$ is the nth state at time $t = 0$. In the Heisenberg representation,

$$i\hbar\frac{da}{dt} = [a, H] = \left[a, \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)\right] = \hbar\omega a. \tag{2.26}$$

The solution is

$$a = a(0)e^{-i\omega t}, \tag{2.27}$$

and similarly

$$a^\dagger = a^\dagger(0)e^{i\omega t}. \tag{2.28}$$

We have shown $a^\dagger a|n\rangle = n|n\rangle$, $a|n\rangle = c_d|n-1\rangle$, and $a^\dagger|n\rangle = c_u|n+1\rangle$. The constants c_d and c_u can be determined by the following equations:

$$\langle n|a^\dagger a|n\rangle = \langle n|n|n\rangle = n, \tag{2.29}$$

$$\langle n|a^\dagger a|n\rangle = \langle n-1|c_d^* c_d|n-1\rangle = |c_d|^2, \tag{2.30}$$

hence $|c_d| = \sqrt{n}$.

$$\langle n|aa^\dagger|n\rangle = \langle n|a^\dagger a + 1|n\rangle = n+1, \tag{2.31}$$

$$\langle n|aa^\dagger|n\rangle = \langle n+1|c_u^* c_u|n+1\rangle = |c_u|^2, \tag{2.32}$$

hence $|c_u| = \sqrt{n+1}$. From Eqs. (2.27) and (2.28), $c_d = |c_d|e^{-i(\omega t-\phi)}$ and $c_u = |c_u|e^{i(\omega t-\phi)}$, where ϕ is a constant phase. The relative phase between $|n\rangle$ and $|n+1\rangle$ can be chosen to be 0, then at $t=0$

$$a|n\rangle = \sqrt{n}|n-1\rangle, \qquad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle. \tag{2.33}$$

Exercise 2.1. Writing $\langle m|n\rangle$ as $\langle 0|a^m a^{\dagger n}|0\rangle$, use Eq. (2.19) to show $\langle m|n\rangle = 0$.

Exercise 2.2. Writing the momentum operator p as $-i\hbar\partial/\partial x$, obtain the ground state wavefunction of the simple harmonic oscillator by solving $a|0\rangle = 0$.

Exercise 2.3. Writing the momentum operator p as $-i\hbar\partial/\partial x$, derive the first excited state wavefunction of the simple harmonic oscillator by applying $a^\dagger$ to $|0\rangle$.

2.4 Quantization of Mechanical Waves

One of the simplest phenomena in systems of many degrees of freedom is mechanical waves. To illustrate how to quantize a mechanical wave, we now consider a thin string of uniform mass and tension distribution. Both ends of the string are fixed, and the length of the string is L. Let us model the string by a sequence of massive beads linked by ideal springs, and denote the displacement of the mth bead from its equilibrium position $u_m(t)$. Assume each bead-spring unit occupies a length a in space. The mass of the bead can be written as ρa, where ρ is the mass density. Similarly, the spring constant can be written as τ/a, where τ is the tension of the string. The Hamiltonian function is just the sum of the kinetic energy of the beads and the potential energy of the springs.

$$H = \frac{\rho}{2} \sum_{m=1}^{N} a \left(\frac{du_m}{dt} \right)^2 + \frac{\tau}{2} \sum_{m=1}^{N} a \left(\frac{u_{m+1} - u_m}{a} \right)^2. \tag{2.34}$$

Now let $a \to 0$ while keeping ρ and τ constant, one has

$$H = \int_L \frac{\rho}{2} \left(\frac{\partial u}{\partial t} \right)^2 + \frac{\tau}{2} \left(\frac{\partial u}{\partial x} \right)^2 dx, \tag{2.35}$$

where we have replaced $u_m(t)$ by $u(x,t)$, the displacement of the string at x. To reduce the Hamiltonian to a much simpler form, we first decompose $u(x,t)$ into Fourier components.

$$u(x,t) = \sqrt{\frac{2}{\rho L}} \sum_m q_m(t) \sin(k_m x), \tag{2.36}$$

where $k_m = n\pi/L$. Then we substitute Eq. (2.36) into Eq. (2.35). The result is

$$H = \frac{1}{2} \sum_m \left(\dot{q}_m^2 + \omega_m^2 q_m^2 \right), \tag{2.37}$$

where $\omega_m \equiv k_m \sqrt{\tau/\rho}$. We see that the Hamiltonian has the same form as a group of independent simple harmonic oscillators, and q_i, $p_j \equiv \dot{q}_j$, are canonical variables. Set

$$a_m \equiv \sqrt{\frac{\omega_m}{2\hbar}} \left(q_m + i\frac{p_m}{\omega_m} \right), \qquad a_m^\dagger \equiv \sqrt{\frac{\omega_m}{2\hbar}} \left(q_m - i\frac{p_m}{\omega_m} \right), \tag{2.38}$$

the Hamiltonian function can be written as

$$H = \sum_m \hbar\omega_m \left(a_m^\dagger a_m + \frac{1}{2} \right), \tag{2.39}$$

where the physical meaning of a_m is the complex amplitude of the mth wave component.

2.5 Quantization of Electromagnetic Waves

The quantization of electromagnetic waves is similar to that of mechanical waves. Consider a region of volume V which contains no charge or electric current. The electromanetic fields can be expressed as superpositions of orthonormal spatial modes. Let us denote $\mathbf{E}_m(\mathbf{r})\exp(-i\omega_m t)$, $-i\mathbf{B}_m(\mathbf{r})\exp(-i\omega_m t)$ the field of the mth space mode, where the real functions $\mathbf{E}_m(\mathbf{r})$ and $\mathbf{B}_m(\mathbf{r})$ are orthonormal complete sets. From the Maxwell equation

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{c\partial t}, \tag{2.40}$$

one has

$$\nabla \times \mathbf{E}_m = \frac{\omega_m}{c}\mathbf{B}_m. \tag{2.41}$$

Let us expand an arbitrary electromagnetic field as

$$\mathbf{E}(\mathbf{r}, t) = \sum_m v_m(t)\mathbf{E}_m(\mathbf{r}), \tag{2.42}$$

$$\mathbf{B}(\mathbf{r}, t) = \sum_m u_m(t)\mathbf{B}_m(\mathbf{r}), \tag{2.43}$$

where $v_m(t)$ and $u_m(t)$ are real functions. Define

$$p_m(t) = -\frac{1}{2}\sqrt{\frac{V}{\pi}}v_m(t), \tag{2.44}$$

$$q_m(t) = \frac{1}{2\omega_m}\sqrt{\frac{V}{\pi}}u_m(t). \tag{2.45}$$

From Eq. (2.40) and (2.41)

$$p_m(t) = \dot{q}_m(t). \tag{2.46}$$

Substitute Eqs. (2.42) and (2.43) into the Hamiltonian function

$$H = \frac{1}{8\pi}\int_V (\mathbf{E}\cdot\mathbf{E} + \mathbf{B}\cdot\mathbf{B})\, d^3r. \tag{2.47}$$

Since $\mathbf{E}_m(\mathbf{r})$ and $\mathbf{B}_m(\mathbf{r})$ are orthonormal, $\int \mathbf{E}_m(\mathbf{r})\cdot\mathbf{E}_n(\mathbf{r})\, d^3r = \delta_{mn}$ and $\int \mathbf{B}_m(\mathbf{r})\cdot\mathbf{B}_n(\mathbf{r})\, d^3r = \delta_{mn}$,

$$H = \frac{1}{2}\sum_m \left(p_m^2 + \omega_m^2 q_m^2\right). \tag{2.48}$$

The Hamiltonian has the same form as a group of independent simple harmonic oscillators. Set

$$a_m \equiv \sqrt{\frac{\omega_m}{2\hbar}}\left(q_m + i\frac{p_m}{\omega_m}\right), \qquad a_m^\dagger \equiv \sqrt{\frac{\omega_m}{2\hbar}}\left(q_m - i\frac{p_m}{\omega_m}\right), \tag{2.49}$$

the Hamiltonian function can be written as

$$H = \sum_m \hbar\omega_m \left(a_m^\dagger a_m + \frac{1}{2} \right). \tag{2.50}$$

Each Fourier component of the electromagnetic field corresponds to an independent simple harmonic oscillator, just like mechanical waves. From (2.49), one has

$$q_m = \sqrt{\frac{\hbar}{2\omega_m}} \left(a_m + a_m^\dagger \right), \qquad p_m = -i\sqrt{\frac{\hbar\omega_m}{2}} \left(a_m - a_m^\dagger \right). \tag{2.51}$$

Substitute into Eqs. (2.44), (2.45), Eqs. (2.42), (2.43) become

$$\mathbf{E}(\mathbf{r}, t) = i \sum_m \sqrt{\frac{2\pi\hbar\omega_m}{V}} \left(a_m - a_m^\dagger \right) \mathbf{E}_m(\mathbf{r}), \tag{2.52}$$

$$\mathbf{B}(\mathbf{r}, t) = \sum_m \sqrt{\frac{2\pi\hbar\omega_m}{V}} \left(a_m + a_m^\dagger \right) \mathbf{B}_m(\mathbf{r}). \tag{2.53}$$

After this quantization procedure, $\mathbf{B}(\mathbf{r}, t)$ and $\mathbf{E}(\mathbf{r}, t)$ are no longer classical variables. They become operators, as they are linear combinations of a_m and $a_m^\dagger$. The mode functions $\mathbf{E}_m(\mathbf{r})$ and $\mathbf{B}_m(\mathbf{r})$ play the role as the coefficients of the linear combinations.

2.6 Coherent States

Since $\langle n|a|n\rangle = \langle n|a^\dagger|n\rangle = 0$, the expectation values of the field operators in a pure n-photon state $|n\rangle$ are always zero. Hence pure n-photon states are not states which correspond to classical electromagnetic waves. It is nature to ask which states give nonzero expectation values of the field operators. A good guess is the eigenstate of a. If $|\alpha\rangle$ is an eigenstate of a_m, since $a_m = a_m(0)e^{-i\omega_m t}$, the eigenvalue must be $\alpha e^{-i\omega_m t}$, where α is a constant. Eqs. (2.52) and (2.53) imply

$$\begin{aligned} \langle\alpha|\mathbf{E}|\alpha\rangle &= i\sqrt{\frac{2\pi\hbar\omega_m}{V}} \mathbf{E}_m(\mathbf{r}) \langle\alpha|a_m - a_m^\dagger|\alpha\rangle \\ &= -2\sqrt{\frac{2\pi\hbar\omega_m}{V}} \mathbf{E}_m(\mathbf{r})|\alpha| \sin(\omega_m t - \phi), \end{aligned} \tag{2.54}$$

$$
\begin{aligned}
\langle\alpha|\mathbf{B}|\alpha\rangle &= \sqrt{\frac{2\pi\hbar\omega_m}{V}}\mathbf{B}_m(\mathbf{r})\,\langle\alpha|a_m + a_m^\dagger|\alpha\rangle \\
&= 2\sqrt{\frac{2\pi\hbar\omega_m}{V}}\mathbf{B}_m(\mathbf{r})|\alpha|\cos(\omega_m t - \phi), \qquad (2.55)
\end{aligned}
$$

where $\phi = \arg(\alpha)$. The expectation value is indeed a nonzero wave. Eqs. (2.54) and (2.55) show that coherent states provide the connection between classical and quantum mechanical descriptions of electromagnetic waves. Similarly, for a simple harmonic oscillator in a coherent state the expectation values of p and q follow their classical trajectory.

$$
\begin{aligned}
\langle\alpha|p|\alpha\rangle &= -i\sqrt{\frac{\hbar m\omega}{2}}\,\langle\alpha|a - a^\dagger|\alpha\rangle \\
&= -\sqrt{2\hbar m\omega}\,|\alpha|\sin(\omega t + \phi), \qquad (2.56)
\end{aligned}
$$

$$
\begin{aligned}
\langle\alpha|q|\alpha\rangle &= \sqrt{\frac{\hbar}{2m\omega}}\,\langle\alpha|a + a^\dagger|\alpha\rangle \\
&= \sqrt{\frac{2\hbar}{m\omega}}\,|\alpha|\cos(\omega t + \phi). \qquad (2.57)
\end{aligned}
$$

Again it is seen the coherent state provides the connection between classical and quantum mechanical descriptions.

To find what $|\alpha\rangle$ is, we expand $|\alpha\rangle$ in a series of $|n\rangle$,

$$|\alpha\rangle = \sum_n c_n|n\rangle. \qquad (2.58)$$

At $t = 0$,

$$\alpha|\alpha\rangle = a|\alpha\rangle = \sum_n c_n\sqrt{n}|n-1\rangle. \qquad (2.59)$$

On the other hand,

$$\alpha|\alpha\rangle = \sum_n \alpha c_n|n\rangle = \sum_n \alpha c_{n-1}|n-1\rangle. \qquad (2.60)$$

Comparing the two equations, one obtains $c_n = (\alpha/\sqrt{n})c_{n-1}$. By induction, $c_n = (\alpha^n/\sqrt{n!})c_0$. Now all the expansion coefficients have been determined except c_0, which can be fixed by the normalization condition $\langle\alpha|\alpha\rangle = |c_0|^2\exp(|\alpha|^2) = 1$. Hence

$$|\alpha\rangle = \exp\left(-\frac{1}{2}|\alpha|^2\right)\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle. \qquad (2.61)$$

The state $|\alpha\rangle$ is called the coherent state.

Exercise 2.4. $|\alpha\rangle$, $|\beta\rangle$ are coherent states. Show that $\langle\alpha|a^\dagger a|\alpha\rangle = |\alpha|^2$, $|\langle\alpha|\beta\rangle|^2 = \exp\left(-|\alpha-\beta|^2\right)$.

Exercise 2.5. Show that $a^\dagger$ has no normalizable eigenvectors.

2.7 Driven Harmonic Oscillators

An illuminating application of the coherent state is the driven harmonic oscillators. Let us consider the Hamiltonian of a simple harmonic oscillator under the influence of an external force $F(t)$:

$$H = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2} - qF(t). \tag{2.62}$$

Eq. (2.62) can be written as:

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) + f(t)\left(a + a^\dagger\right), \tag{2.63}$$

where $f(t) = -\sqrt{\hbar/2m\omega}F(t)$. Under the time dependent Hamiltonian, the harmonic oscillator evolves from the ground state to an unknown final state $|f\rangle$.

$$|f\rangle = U(t)|0\rangle, \tag{2.64}$$

where $U(t)$ is the time evolution operator. We may expand the final state into the number states $|n\rangle$, the expansion coefficients are

$$c_n = \langle n|U(t)|0\rangle = \frac{1}{\sqrt{n!}}\langle 0|a^n U(t)|0\rangle. \tag{2.65}$$

Since $U(t)U(-t) = 1$, Eq. (2.65) can be written in the following form:

$$c_n = \frac{1}{\sqrt{n!}}\langle 0|U(t)[U(-t)aU(t)]^n|0\rangle. \tag{2.66}$$

For simplicity let us define $U(-t)aU(t) \equiv a(t)$. From

$$\frac{dU(t)}{dt} = -\frac{i}{\hbar}HU(t),$$

the equation of motion for the operator $a(t)$ is

$$\frac{da(t)}{dt} = \frac{i}{\hbar}U(-t)(Ha - aH)U(t). \tag{2.67}$$

Since

$$Ha - aH = -\hbar\omega a - f(t),$$

$$\frac{da(t)}{dt} = -i\omega a(t) - \frac{i}{\hbar} f(t). \tag{2.68}$$

Writing the operator a in the slow varying amplitude form $a(t) = \tilde{a}(t)e^{-i\omega t}$, it is not difficult to find that the solution of Eq. (2.68) is

$$\tilde{a}(t) = \tilde{a}(0) - \frac{i}{\hbar} g(\omega), \tag{2.69}$$

where $\tilde{a}(0) = a(0) = a$ and

$$g(\omega) = \int_0^t e^{i\omega s} f(s)\, ds \tag{2.70}$$

is the spectral content of $f(t)$ at ω. Hence

$$a(t) = \left[a - \frac{i}{\hbar} g(\omega)\right] e^{-i\omega t}. \tag{2.71}$$

Substitute Eq. (2.71) into Eq. (2.66) we obtain

$$c_n = \frac{1}{\sqrt{n!}} \langle 0|U(t) \left\{ \left[a - \frac{i}{\hbar} g(\omega)\right] e^{-i\omega t} \right\}^n |0\rangle = \frac{\alpha^n}{\sqrt{n!}} \langle 0|U(t)|0\rangle, \tag{2.72}$$

where $\alpha = (-i/\hbar)g(\omega)\exp(-i\omega t)$. The constant $\langle 0|U(t)|0\rangle$ can be evaluated from the normalization condition to be $\exp\left(-|\alpha|^2/2\right)$. Eq. (2.72) shows that the final state is a coherent state.

2.8 Classical Electromagnetic Waves

We have shown that in the limit of large photon number, quantum coherent states correspond to classical electromagnetic waves. To further clarify the relation between coherent states and classical electromagnetic waves, let us examine quantum mechanically the most common processes of generating electromagnetic waves: radiation from an oscillating current distribution in an antenna.

The Hamiltonian function of a group of electrons in electromagnetic fields is [3]:

$$H = \sum_i \frac{(\mathbf{p}_i - e\mathbf{A}/c)^2}{2m} + V(\mathbf{r}_1, \mathbf{r}_2, \ldots) + \frac{1}{8\pi} \int\limits_V \left(|\mathbf{E}|^2 + |\mathbf{B}|^2\right) d^3r, \tag{2.73}$$

where the summation is over all the electrons. The Maxwell equations and the Lorentz force equation can be derived from this Hamiltonian function. In the radio frequency region, the energy of the photon is small, the radiation process does not affect much the wavefunction of the electrons. Further more, the length scale we are interested in, the wavelength of the photons, is much longer than the de Broglie wavelength of the electrons. In this case we can treat the electrons classically and replace the momentum operator by its expectation value. Eq. (2.73) can then be simplified:

$$H = \frac{1}{8\pi} \int_V \left(|\mathbf{E}|^2 + |\mathbf{B}|^2 \right) d^3r - \frac{1}{c} \int \mathbf{j}(\mathbf{r}, t) \cdot \mathbf{A}(\mathbf{r}, t)\, d^3r. \tag{2.74}$$

Where in Eq. (2.74) we have ignored the small $(e^2/2mc^2)\mathbf{A} \cdot \mathbf{A}$ term in Eq. (2.73), and $\mathbf{j}(\mathbf{r}, t)$ is the current distribution defined by

$$\mathbf{j}(\mathbf{r}, t) = \sum_i \frac{e\mathbf{p}_i}{m} \delta(\mathbf{r} - \mathbf{r}_i). \tag{2.75}$$

When the electron density is large, the uncertainty or fluctuation of $\mathbf{j}(\mathbf{r}, t)$ is small as a consequence of the central limit theorem. We can then treat $\mathbf{j}(\mathbf{r}, t)$ as a known parameter. Since $\mathbf{E} = -(1/c)(\partial\mathbf{A}/\partial t)$, from Eq. (2.42), one obtains

$$\begin{aligned} \mathbf{A}(\mathbf{r}, t) &= 2c\sqrt{\frac{\pi}{V}} \sum_m q_m(t) \mathbf{E}_m(\mathbf{r}) \\ &= c \sum_m \sqrt{\frac{2\pi\hbar}{V\omega_m}} \left(a_m + a_m^\dagger \right) \mathbf{E}_m(\mathbf{r}). \end{aligned} \tag{2.76}$$

Define

$$j_m(t) = \sqrt{\frac{2\pi\hbar}{V\omega_m}} \int \mathbf{j}(\mathbf{r}, t) \cdot \mathbf{E}_m(\mathbf{r})\, d^3r, \tag{2.77}$$

Eq. (2.74) can be written as

$$H = \sum_m \hbar\omega_m \left(a_m^\dagger a_m + \frac{1}{2} \right) - \sum_m j_m(t) \left(a_m + a_m^\dagger \right). \tag{2.78}$$

Eq. (2.78) is of the same form as Eq. (2.63). Thus from the result of section 2.7, we arrive at the conclusion that the radiation from an oscillating current distribution is a coherent state.

Exercise 2.6. The fractional uncertainty of the electric field $\mathbf{E}$ is defined by

$$\frac{\Delta E}{E} = \sqrt{\frac{|\langle\alpha|\mathbf{E} \cdot \mathbf{E}|\alpha\rangle|_{\langle t\rangle} - |\langle\alpha|\mathbf{E}|\alpha\rangle|^2_{\langle t\rangle}}{|\langle\alpha|\mathbf{E}|\alpha\rangle|^2_{\langle t\rangle}}}, \tag{2.79}$$

where $\langle t \rangle$ represents averaging over one cycle. Show that the electric field, 10 km away from an FM station running at 100,000 watts, has practically no uncertainty. In this case the classical electromagnetic theory is as good as the quantum one.

Solution: According to Eqs. (2.52), (2.54), if we define

$$\mathbf{C}_m = \sqrt{\frac{2\pi\hbar\omega_m}{V}}\mathbf{E}_m(\mathbf{r}),$$

we have

$$\mathbf{E} = i\mathbf{C}_m\left(a_m - a_m^\dagger\right),$$

$$\langle\alpha|\mathbf{E}|\alpha\rangle = -2\mathbf{C}_m|\alpha|\sin(\omega t - \phi),$$

$$|\langle\alpha|\mathbf{E}|\alpha\rangle|^2 = 4|\mathbf{C}_m|^2|\alpha|^2\sin^2(\omega t - \phi),$$

$$\begin{aligned}\mathbf{E}\cdot\mathbf{E} &= |\mathbf{C}_m|^2\left\{-a_m^2 - a_m^{\dagger 2} + a_m a_m^\dagger + a_m^\dagger a_m\right\}\\ &= |\mathbf{C}_m|^2\left\{-a_m^2 - a_m^{\dagger 2} + 1 + 2a_m^\dagger a_m\right\},\end{aligned}$$

$$\langle\alpha|\mathbf{E}\cdot\mathbf{E}|\alpha\rangle = |\mathbf{C}_m|^2\left\{-2|\alpha|^2\cos[2(\omega_m t - \phi)] + 1 + 2\,|\alpha|^2\right\}.$$

Averaging over one cycle, one obtains

$$|\langle\alpha|\mathbf{E}|\alpha\rangle|^2_{\langle t\rangle} = 2\,|\mathbf{C}_m|^2\,|\alpha|^2,$$

$$|\langle\alpha|\mathbf{E}\cdot\mathbf{E}|\alpha\rangle|_{\langle t\rangle} = |\mathbf{C}_m|^2\left(1 + 2\,|\alpha|^2\right).$$

Therefore

$$|\langle\alpha|\mathbf{E}\cdot\mathbf{E}|\alpha\rangle|_{\langle t\rangle} - |\langle\alpha|\mathbf{E}|\alpha\rangle|^2_{\langle t\rangle} = |\mathbf{C}_m|^2,$$

and

$$\frac{\Delta E}{E} = \frac{1}{\sqrt{2}\,|\alpha|}.$$

Now we shall calculate the value of $|\alpha|$. Consider the energy density ρ.

$$\rho = \frac{\text{power}}{4\pi r^2 c} = \frac{10^5}{4\pi\times(10^4)^2\times 3\times 10^8}.$$

Alternatively, ρ can also be calculated the following way:

$$\rho = \frac{\text{energy per mode}}{\text{volume per mode}},$$

where

$$\text{volume per mode} = \frac{\text{volume}}{\text{number of modes}} = \left(\frac{\lambda}{2}\right)^3,$$

and

$$\text{energy per mode} = \frac{1}{4\pi}\int_V |\langle\alpha|\mathbf{E}|\alpha\rangle|^2_{\langle t\rangle}\, d^3v = \frac{V}{4\pi}\cdot 2\,|\mathbf{C}_m|^2\,|\alpha|^2\,.$$

Therefore

$$\rho = \frac{V}{2\pi}\,|\mathbf{C}_m|^2\,|\alpha|^2\left(\frac{2}{\lambda}\right)^3. \tag{2.80}$$

Because $\mathbf{C}_m = \sqrt{2\pi\hbar\omega_m/V}\,\mathbf{E}_m(\mathbf{r})$,

$$\rho = h\nu\,|\alpha|^2\left(\frac{2}{\lambda}\right)^3. \tag{2.81}$$

Hence

$$\begin{aligned}|\alpha| &= \sqrt{\frac{\rho\lambda^3}{8h\nu}} \approx 5.4\times 10^6,\\ \frac{\Delta E}{E} &= \frac{1}{\sqrt{2}\,|\alpha|} \approx 1.9\times 10^{-7}.\end{aligned}$$

It is seen from Eq. (2.81) that the quantum nature of electromagnetic waves become negligible when the photon density is much greater than $(2/\lambda)^3$.

Exercise 2.7. Calculate the fractional uncertainty $\Delta E/E$ in Exercise 2.7 for a 100 watts monochromatic lamp (wavelength 600 nm) at 1 km away.

Exercise 2.8. When the photoelectric effect was first discovered, there was no quantum mechanics. Physicists then were puzzled by the fast response of this effect. Even if the intensity of the incident light was reduced so much that it would take a long time to accumulate enough energy to kick out the electron, fast response was still observed. Low input intensity only reduces the probability of the electron being kicked out. Can you explain it by calculating the fractional uncertainty of the Poynting vector $\mathbf{S} = (c/4\pi)\mathbf{E}\times\mathbf{B}$?

Exercise 2.9. A simple harmonic oscillator in its ground state is suddenly displaced in space by d. Namely $\psi(x,0)$, the wavefunction at $t=0$, is changed from $|0\rangle \propto \exp\left[-m\omega x^2/(2\hbar)\right]$ to $|\tilde{0}\rangle \propto \exp\left[-m\omega(x-d)^2/(2\hbar)\right]$. What is the probability amplitude c_m of being in state $|n\rangle$? Show that

$$\begin{aligned}\psi(x,t) &= \sqrt[4]{\frac{m\omega}{\pi\hbar}}\exp\left[-\frac{m\omega}{2\hbar}(x-d\cos\omega t)^2\right]\times\\ &\qquad \exp\left[i\left(-\frac{\omega t}{2}-\frac{m\omega}{\hbar}xd\sin\omega t+\frac{m\omega}{4\hbar}d^2\sin 2\omega t\right)\right].\end{aligned} \tag{2.82}$$

Solution: Change the operator $x - d$ to $\tilde{x}$. The annihilation operator a changes accordingly to $\tilde{a} = a - \sqrt{m\omega/(2\hbar)}\, d$.

$$\begin{aligned} c_n(t) &= \langle n|U(t)|\tilde{0}\rangle \\ &= \frac{1}{\sqrt{n!}}\langle 0|a^n U(t)|\tilde{0}\rangle \\ &= \frac{1}{\sqrt{n!}}\langle 0|U(t)[U(-t)aU(t)]^n|\tilde{0}\rangle \end{aligned} \tag{2.83}$$

Define $a(t) \equiv U(-t)aU(t)$, from $\dot{U} = -(i/\hbar)HU$ and $H = \hbar\omega(a^\dagger a + \frac{1}{2})$ one obtains $\dot{a}(t) = -i\omega a(t)$, hence $a(t) = a\exp(-i\omega t)$. Substitute it into Eq. (2.83),

$$\begin{aligned} c_n(t) &= \frac{1}{\sqrt{n!}}\langle 0|U(t)[a\exp(-i\omega t)]^n|\tilde{0}\rangle \\ &= \frac{1}{\sqrt{n!}}\langle 0|U(t)\left[\left(\tilde{a} + \sqrt{\frac{m\omega}{2\hbar}}d\right)\exp(-i\omega t)\right]^n|\tilde{0}\rangle \\ &= \frac{1}{\sqrt{n!}}\left[\sqrt{\frac{m\omega}{2\hbar}}d\,\exp(-i\omega t)\right]^n\langle 0|U(t)|\tilde{0}\rangle. \end{aligned} \tag{2.84}$$

Since $\sum_n |c_n(t)|^2 = 1$,

$$|\langle 0|U(t)|\tilde{0}\rangle| = \exp\left(-\frac{m\omega}{4\hbar}d^2\right). \tag{2.85}$$

Therefore

$$c_n(t) = \exp\left(-\frac{i\omega t}{2} - \frac{m\omega}{4\hbar}d^2\right)\frac{1}{\sqrt{n!}}\left[\sqrt{\frac{m\omega}{2\hbar}}d\,\exp(-i\omega t)\right]^n, \tag{2.86}$$

where we have added the factor $\exp(-i\omega t/2)$ to account for the $\hbar\omega/2$ in the Hamiltonian.

$$\begin{aligned} \psi(x,t) &= \sum_n c_n(t)|n\rangle \\ &= \exp\left(-\frac{i\omega t}{2} - \frac{m\omega}{4\hbar}d^2\right)\sum_n \frac{1}{\sqrt{n!}}\left[\sqrt{\frac{m\omega}{2\hbar}}d\,\exp(-i\omega t)\right]^n|n\rangle. \end{aligned} \tag{2.87}$$

Because

$$\begin{aligned} \psi(x,0) &= \exp\left(-\frac{m\omega}{4\hbar}d^2\right)\sum_n \frac{1}{\sqrt{n!}}\left(\sqrt{\frac{m\omega}{2\hbar}}d\right)^n|n\rangle \\ &= \sqrt[4]{\frac{m\omega}{\pi\hbar}}\exp\left[-\frac{m\omega}{2\hbar}(x-d)^2\right], \end{aligned} \tag{2.88}$$

the sum on the right hand side of Eq. (2.87) can be put into a closed form by replacing d in Eq. (2.88) by $d\exp(-i\omega t)$,

$$\begin{aligned}\psi(x,t) &= \exp\left(-\frac{i\omega t}{2}-\frac{m\omega}{4\hbar}d^2\right)\exp\left[\frac{m\omega}{4\hbar}d^2\exp(-2i\omega t)\right]\times \\ &\qquad \sqrt[4]{\frac{m\omega}{\pi\hbar}}\exp\left\{-\frac{m\omega}{2\hbar}[x-d\exp(-i\omega t)]^2\right\} \\ &= \sqrt[4]{\frac{m\omega}{\pi\hbar}}\exp\left[-\frac{m\omega}{2\hbar}(x-d\cos\omega t)^2\right]\times \\ &\qquad \exp\left[i\left(-\frac{\omega t}{2}-\frac{m\omega}{\hbar}xd\sin\omega t+\frac{m\omega}{4\hbar}d^2\sin 2\omega t\right)\right]. \qquad (2.89)\end{aligned}$$

Exercise 2.10. The Green function solution $K(x,x';t)$ of the Schrödinger equation is defined by

$$\psi(x,t)=\int K(x,x';t)\psi(x',0)\,dx' \qquad (2.90)$$

for any wavefunction $\psi(x,t)$ and its initial condition $\psi(x,0)$. Show that for the simple harmonic oscillator

$$K(x,x';t)=\left(\frac{m\omega}{2\pi i\hbar\sin\omega t}\right)^{1/2}\exp\left[-\frac{m\omega}{2i\hbar\sin\omega t}(x^2\cos\omega t-2xx'+x'^2\cos\omega t)\right]. \qquad (2.91)$$

Solution: We first find a $K(x,x';t)$ which satisfies Eq. (2.90) for the particular initial condition in Exercise 2.10. Namely find a $K(x,x';t)$ satisfying

$$\begin{aligned}\int K(x,x';t)\exp\left[-\frac{m\omega}{2\hbar}(x'-d)^2\right]dx' &= \exp\left(-\frac{i\omega t}{2}\right)\times \\ \exp\left\{-\frac{m\omega}{2\hbar}\left[x^2-2xd\exp(-i\omega t)+d^2\exp(-i\omega t)\cos\omega t\right]\right\}&. \qquad (2.92)\end{aligned}$$

As an educated guess, we assume

$$K(x,x';t)=N\exp(fx^2+gxx'+rx'^2), \qquad (2.93)$$

where N, f, g, and r are functions of time.

$$\begin{aligned}\int K(x,x';t)\exp\left[-\frac{m\omega}{2\hbar}(x'-d)^2\right]dx' &= N\sqrt{\frac{2\hbar\pi}{m\omega-2\hbar r}}\times \\ \exp\left[\frac{(\hbar g^2/2+m\omega f-2\hbar fr)x^2+(m\omega g)xd+(m\omega r)d^2}{m\omega-2\hbar r}\right]&. \qquad (2.94)\end{aligned}$$

Comparing Eqs. (2.92) and (2.94), one obtains

$$N\sqrt{\frac{2\hbar\pi}{m\omega - 2\hbar r}} = \exp\left(\frac{-i\omega t}{2}\right), \tag{2.95}$$

$$\frac{\hbar g^2/2 + m\omega f - 2\hbar f r}{m\omega - 2\hbar r} = -\frac{m\omega}{2\hbar}, \tag{2.96}$$

$$\frac{m\omega g}{m\omega - 2\hbar r} = \frac{m\omega}{\hbar}\exp(-i\omega t), \tag{2.97}$$

$$\frac{m\omega r}{m\omega - 2\hbar r} = -\frac{m\omega}{2\hbar}\exp(-i\omega t)\cos\omega t. \tag{2.98}$$

The solution is

$$\begin{aligned} r &= -\frac{m\omega\cos\omega t}{2i\hbar\sin\omega t} \\ g &= \frac{m\omega}{i\hbar\sin\omega t} \\ f &= -\frac{m\omega\cos\omega t}{2i\hbar\sin\omega t} \\ N &= \left(\frac{m\omega}{2\pi i\hbar\sin\omega t}\right)^{1/2}. \end{aligned} \tag{2.99}$$

That is

$$K(x,x';t) = \left(\frac{m\omega}{2\pi i\hbar\sin\omega t}\right)^{1/2}\exp\left[-\frac{m\omega}{2i\hbar\sin\omega t}(x^2\cos\omega t - 2xx' + x'^2\cos\omega t)\right]. \tag{2.100}$$

If it is proved that for all n,

$$\psi_n(x,t) = \int K(x',x;t)\psi_n(x',0)\,dx', \tag{2.101}$$

where $\psi_m(x,t)$ is the nth eigenstate, then Eq. (2.90) holds for any arbitrary $\psi(x,t)$ and its initial condition $\psi(x,0)$. We shall prove it by induction. For $n=0$,

$$\begin{aligned} \int K(x',x;t)\psi_0(x',0)\,dx' &= \int K(x',x;t)\sqrt[4]{\frac{m\omega}{\pi\hbar}}\exp\left(-\frac{m\omega}{2\hbar}x'^2\right)dx' \\ &= \exp\left(-\frac{i\omega t}{2}\right)\sqrt[4]{\frac{m\omega}{\pi\hbar}}\exp\left(-\frac{m\omega}{2\hbar}x^2\right) \\ &= \psi_0(x,t). \end{aligned} \tag{2.102}$$

Assume that Eq. (2.101) holds for $n = k$, then

$$
\begin{aligned}
\int & K(x, x'; t)\psi_{k+1}(x', 0)\, dx' \\
&= \frac{1}{\sqrt{k+1}} \int K(x, x'; t) \left[\sqrt{\frac{m\omega}{2\hbar}} \left(x' - \frac{\hbar}{m\omega} \frac{\partial}{\partial x'} \right) \psi_k(x', 0) \right] dx' \\
&= \frac{1}{\sqrt{k+1}} \int \left[\sqrt{\frac{m\omega}{2\hbar}} \left(x' + \frac{\hbar}{m\omega} \frac{\partial}{\partial x'} \right) K(x, x'; t) \right] \psi_k(x', 0)\, dx' \\
&= \frac{\exp(-i\omega t)}{\sqrt{k+1}} \sqrt{\frac{m\omega}{2\hbar}} \left(x - \frac{\hbar}{m\omega} \frac{\partial}{\partial x} \right) \int K(x, x'; t)\psi_k(x', 0)\, dx' \\
&= \frac{\exp(-i\omega t)}{\sqrt{k+1}} \sqrt{\frac{m\omega}{2\hbar}} \left(x - \frac{\hbar}{m\omega} \frac{\partial}{\partial x} \right) \psi_k(x, t) \\
&= \psi_{k+1}(x, t),
\end{aligned}
\tag{2.103}
$$

namely, Eq. (2.101) holds for $n = k + 1$.

Chapter 3

Interaction between Radiation and Matter

3.1 Fermi's Golden Rule

Before we look into the interaction between radiation and matter with quantum mechanics, let us derive an important formula that is commonly used in first-order time-dependent perturbation. Consider a general Hamiltonian $H_0 + H_I$, where H_0 is the unperturbed Hamiltonian whose eigenfunctions are known, and H_I is the interaction Hamiltonian whose magnitude is much smaller than H_0. Assume $u_k(x)$ are the eigenfunctions of H_0, a general wavefunction of H can be expanded as

$$\psi(x,t) = \sum_k c_k(t) u_k(x) e^{-iE_k t/\hbar}. \tag{3.1}$$

The equation of motion is

$$\begin{aligned}(H_0 + H_I)\psi(x,t) &= i\hbar \frac{\partial \psi}{\partial t} \\ &= i\hbar \sum_k \left[\dot{c}_k u_k(x) e^{-iE_k t/\hbar} - i\frac{E_k}{\hbar} c_k u_k(x) e^{-iE_k t/\hbar} \right]\end{aligned} \tag{3.2}$$

Because $u_k(x)$ is an eigenfunction of H_0 with eigenenergy E_k, one has

$$\sum_k H_I c_k u_k(x) e^{-iE_k t/\hbar} = i\hbar \sum_k \dot{c}_k u_k(x) e^{-iE_k t/\hbar}. \tag{3.3}$$

Multiplying both sides by $u_m^* e^{iE_m t/\hbar}$ and integrate over x, one obtains the differential equation

$$\dot{c}_m = \frac{1}{i\hbar}\sum_k \langle m|H_I|k\rangle e^{i(E_m-E_k)t/\hbar} c_k(t). \tag{3.4}$$

Suppose only state l is populated when the perturbation is turned on at $t = 0$, *i.e.* $c_k(0) = \delta_{kl}$, for small t one may approximate c_m by

$$c_m(t) = \frac{1}{i\hbar}\int_0^t \langle m|H_I|l\rangle e^{i(E_m-E_l)t'/\hbar}\, dt'. \tag{3.5}$$

If H_I is time-independent, the integration can be carried out exactly. One has

$$\begin{aligned} |c_m(t)|^2 &= |\langle m|H_I|l\rangle|^2 \frac{|1-e^{i(E_m-E_l)t/\hbar}|^2}{(E_m-E_l)^2} \\ &= |\langle m|H_I|l\rangle|^2 \frac{2-2\cos[(E_m-E_l)t/\hbar]}{(E_m-E_l)^2} \\ &= |\langle m|H_I|l\rangle|^2 \frac{4\sin^2[(E_m-E_l)t/(2\hbar)]}{(E_m-E_l)^2}. \end{aligned} \tag{3.6}$$

For $t \gg \hbar/(E_m - E_l)$, Eq. (3.6) can be simplified to

$$|c_m(t)|^2 = \frac{2\pi}{\hbar}|\langle m|H_I|l\rangle|^2\, t\, \delta(E_m - E_l), \tag{3.7}$$

because

$$\lim_{\alpha\to\infty} \frac{1}{\pi}\frac{\sin^2(\alpha x)}{\alpha x^2} = \delta(x). \tag{3.8}$$

The transition rate from state l to state m is therefore

$$W_{l\to m} \equiv \frac{|c_m(t)|^2}{t} = \frac{2\pi}{\hbar}|\langle m|H_I|l\rangle|^2\delta(E_m - E_l). \tag{3.9}$$

This is known as Fermi's golden rule.

3.2 Emission and Absorption of Photons by Atoms

The Hamiltonian function of a charged particle in electromagnetic fields is [3]:

$$H = \frac{1}{2m}\left(\mathbf{p} - \frac{e\mathbf{A}}{c}\right)^2 + V(\mathbf{r}) + \frac{1}{8\pi}\int_V \left(|\mathbf{E}|^2 + |\mathbf{B}|^2\right) d^3r. \tag{3.10}$$

The coupling between the charged particle and the field is expressed by the interaction part of the Hamiltonian

$$H_{\text{int}} = \frac{-e}{2mc}\left(\mathbf{p}\cdot\mathbf{A} + \mathbf{A}\cdot\mathbf{p}\right) + \frac{e^2}{2mc^2}\mathbf{A}\cdot\mathbf{A}. \tag{3.11}$$

When the field is weak, the interaction Hamiltonian can be treated as a perturbation of the decoupled system. By Fermi's golden rule, if the atom absorbs a photon and makes a transition from state A to B, the transition rate is

$$W_{A\to B} = \frac{2\pi}{\hbar}\left|\langle B|\langle n_m - 1|H_{\text{int}}|n_m\rangle|A\rangle\right|^2 \delta\left(E_f - E_i\right), \tag{3.12}$$

where we have expressed the state $|i\rangle$ and $|f\rangle$ as the product of the atomic states $|A\rangle$, $|B\rangle$ and the photonic states $|n_m\rangle$, $|n_m - 1\rangle$. If we ignore the $\mathbf{A}\cdot\mathbf{A}$ term, then from Eq. (2.76)

$$\begin{aligned} H_{\text{int}} &\approx \frac{-e}{2mc}\left(\mathbf{p}\cdot\mathbf{A} + \mathbf{A}\cdot\mathbf{p}\right) \\ &= -\sqrt{\frac{e^2\pi\hbar}{2m^2\omega_m V}}\Big[\mathbf{p}\cdot\mathbf{E}_m(\mathbf{r})\left(a_m + a_m^\dagger\right) + \\ &\qquad\qquad \mathbf{E}_m(\mathbf{r})\cdot\mathbf{p}\left(a_m + a_m^\dagger\right)\Big]. \end{aligned} \tag{3.13}$$

Note that $\mathbf{p}$ and $\mathbf{r}$ operate on the atomic states, and a_m and $a_m^\dagger$ operate on the photonic states. Because $a_m|n_m\rangle = \sqrt{n_m}\,|n_m - 1\rangle$ and $a_m^\dagger|n_m\rangle = \sqrt{n_m + 1}\,|n_m + 1\rangle$,

$$W_{A\to B} = \frac{e^2\pi^2 n_m}{m^2\omega_m V}\left|\langle B|\mathbf{p}\cdot\mathbf{E}_m(\mathbf{r}) + \mathbf{E}_m(\mathbf{r})\cdot\mathbf{p}|A\rangle\right|^2 \delta\left(E_f - E_i\right), \tag{3.14}$$

Similarly, if the atom emits a photon and makes a transition from state B to A, then the transition rate is

$$\begin{aligned} W_{B\to A} &= \frac{2\pi}{\hbar}\left|\langle A|\langle n_m + 1|H_{\text{int}}|n_m\rangle|B\rangle\right|^2 \delta\left(E_f - E_i\right) \\ &= \frac{e^2\pi^2(n_m + 1)}{m^2\omega_m V}\left|\langle A|\mathbf{p}\cdot\mathbf{E}_m(\mathbf{r}) + \mathbf{E}_m(\mathbf{r})\cdot\mathbf{p}|B\rangle\right|^2 \delta\left(E_f - E_i\right) \end{aligned} \tag{3.15}$$

In deriving Eqs. (3.14) and (3.15), we have ignored the $\mathbf{A}\cdot\mathbf{A}$ term in Eq. (3.11) because of the weak field assumption. At large photon number, the probabilities of absorption and emission is the same, whereas at zero photon number, the probability is zero for absorption and nonzero for emission. The phenomena of emission at zero input, called spontaneous emission, can not be explained by classical theories. It accounts to the first order for the finite life time of the excited states of atoms.

Exercise 3.1. Calculate the ratio of $|e\mathbf{A}/c|$ and $|\mathbf{p}|$ for the following two cases: (1)100-watt monochromatic lamp (wavelength 600 nm) at one meter away, (2)100-fs laser pulses (center wavelength 800 nm) with 1-mJ pulse energy focused to 10-μm diameter.

Exercise 3.2. Estimate in order of magnitude the life time of the $2p$ state of the hydrogen atom.

3.3 Multipole Expansion of the Radiation Interaction

When an atom makes a transition and absorbs or emits a photon, the energy of the states and the photon energy must satisfy the energy conservation relation

$$E_f^{\text{atom}} - E_i^{\text{atom}} = \hbar\omega, \tag{3.16}$$

as expressed in the δ-function in Eq. (3.14) and Eq. (3.15). In the optical region, the wavelength of the photon which satisfies Eq. (3.16) is much larger than the atomic radius. In other words, $\mathbf{E}_m(\mathbf{r})$ is approximately equal to a constant vector $\mathbf{E}_m$, and

$$H_{\text{int}} \approx \frac{-e}{mc}\mathbf{p}\cdot\mathbf{A} = c_m \mathbf{p}\cdot\mathbf{E}_m \left(a_m + a_m^\dagger\right), \tag{3.17}$$

where $c_m \equiv -(1/m)\sqrt{2e^2\pi\hbar/(\omega_m V)}$. Therefore

$$W_{A\to B} \approx \frac{4e^2\pi^2 n_m}{m^2\omega_m V} \left|\langle B|\mathbf{p}\cdot\mathbf{E}_m|A\rangle\right|^2 \delta\left(E_f - E_i\right), \tag{3.18}$$

$$W_{B\to A} \approx \frac{4e^2\pi^2 (n_m+1)}{m^2\omega_m V} \left|\langle A|\mathbf{p}\cdot\mathbf{E}_m|B\rangle\right|^2 \delta\left(E_f - E_i\right). \tag{3.19}$$

In the following we want to show that

$$\frac{-e}{mc}\mathbf{p}\cdot\mathbf{A} = \frac{e}{c}\mathbf{r}\cdot\frac{\partial \mathbf{A}}{\partial t}.$$

Since

$$\frac{e}{c}\mathbf{r}\cdot\frac{\partial \mathbf{A}}{\partial t} = -mc_m \mathbf{r}\cdot\mathbf{E}_m\left[(-i\omega_m)a_m + (i\omega_m)a_m^\dagger\right], \tag{3.20}$$

we have

$$\begin{aligned}
&\langle B|\langle n_m - 1|\frac{e}{c}\mathbf{r}\cdot\frac{\partial\mathbf{A}}{\partial t}|n_m\rangle|A\rangle \\
&\qquad = c_m(im\omega_m)\langle B|\langle n_m - 1|\mathbf{r}\cdot\mathbf{E}_m a_m|n_m\rangle|A\rangle, \qquad (3.21)\\
&\langle A|\langle n_m + 1|\frac{e}{c}\mathbf{r}\cdot\frac{\partial\mathbf{A}}{\partial t}|n_m\rangle|B\rangle \\
&\qquad = c_m(-im\omega_m)\langle A|\langle n_m + 1|\mathbf{r}\cdot\mathbf{E}_m a_m^{\dagger}|n_m\rangle|B\rangle. \qquad (3.22)
\end{aligned}$$

Moreover, from

$$[H_{\text{atom}}, \mathbf{r}] = \left[\frac{\mathbf{p}^2}{2m}, \mathbf{r}\right] = -\frac{i\hbar}{m}\mathbf{p}, \qquad (3.23)$$

one has

$$\begin{aligned}
\langle A|\mathbf{p}|B\rangle &= \langle A|\frac{im}{\hbar}[H_{\text{atom}}, \mathbf{r}]|B\rangle = -\frac{im(E_B - E_A)}{\hbar}\langle A|\mathbf{r}|B\rangle \\
&= -im\omega_m\langle A|\mathbf{r}|B\rangle, \qquad (3.24)
\end{aligned}$$

and similarly

$$\langle B|\mathbf{p}|A\rangle = im\omega_m\langle B|\mathbf{r}|A\rangle. \qquad (3.25)$$

Then, from Eqs. (3.17) and (3.24), (3.25),

$$\begin{aligned}
&\langle B|\langle n_m - 1|\frac{-e}{mc}\mathbf{p}\cdot\mathbf{A}|n_m\rangle|A\rangle \\
&\qquad = c_m\langle B|\langle n_m - 1|\mathbf{p}\cdot\mathbf{E}_m a_m|n_m\rangle|A\rangle \\
&\qquad = c_m(im\omega_m)\langle B|\langle n_m - 1|\mathbf{r}\cdot\mathbf{E}_m a_m|n_m\rangle|A\rangle, \qquad (3.26)\\
&\langle A|\langle n_m + 1|\frac{-e}{mc}\mathbf{p}\cdot\mathbf{A}|n_m\rangle|B\rangle \\
&\qquad = c_m\langle A|\langle n_m + 1|\mathbf{p}\cdot\mathbf{E}_m a_m^{\dagger}|n_m\rangle|B\rangle \\
&\qquad = c_m(-im\omega_m)\langle A|\langle n_m + 1|\mathbf{r}\cdot\mathbf{E}_m a_m^{\dagger}|n_m\rangle|B\rangle, \qquad (3.27)
\end{aligned}$$

We have proved

$$\frac{-e}{mc}\mathbf{p}\cdot\mathbf{A} = \frac{e}{c}\mathbf{r}\cdot\frac{\partial\mathbf{A}}{\partial t} = -e\mathbf{r}\cdot\mathbf{E}.$$

This is called the dipole approximation. This process can be continued by taking more and more terms in the expansion

$$\mathbf{E}_m(\mathbf{x}) = \mathbf{E}_m(\mathbf{0}) + \frac{\partial\mathbf{E}_m}{\partial x^\mu}(\mathbf{0})dx^\mu + \frac{1}{2}\frac{\partial^2\mathbf{E}_m}{\partial x^\mu\partial x^\nu}(\mathbf{0})dx^\mu dx^\nu + \cdots.$$

It is called the multipole expansion.

Chapter 4

Amplification of Light

4.1 Density Matrix

In this chapter we shall use a simple model to describe the resonant interaction between radiation and a large number of atoms or molecules. It will be shown that if the states of the atoms or molecules are properly prepared, light can be amplified in such media.

As one would expect, we may derive the interaction with a group of atoms or molecules by averaging over the group. Density matrix is a standard formalism in quantum statistical mechanics for describing the ensemble average of physical quantities. In quantum mechanics a general state $|s\rangle$ can be represented by its expansion coefficients in the Hilbert space formed by a complete set of orthogonal bases.

$$|s\rangle = \sum_i c_i |i\rangle. \tag{4.1}$$

The coefficients c_i, called probability amplitudes, are related to the physical probability of being in state $|i\rangle$.

$$\text{probability in state } |i\rangle = |c_i|^2 . \tag{4.2}$$

Because c_i itself is not the probability, it can not be averaged statistically. If we expand the expectation value of an operator A, instead of the state $|s\rangle$, in terms of c_i, then

$$\langle A\rangle \equiv \langle s|A|s\rangle = \sum_{m,n} c_m^* c_n A_{mn}, \tag{4.3}$$

where $A_{mn} = \langle m|A|n\rangle$. The ensemble average can be written conveniently as

$$\overline{\langle A\rangle} = \sum_{m,n} \overline{c_m^* c_n} A_{mn}. \tag{4.4}$$

Now we can define the density matrix ρ by

$$\rho_{nm} \equiv \overline{c_m^* c_n}, \tag{4.5}$$

then

$$\overline{\langle A\rangle} = \sum_{m,n} \rho_{nm} A_{mn} = \mathrm{tr}(\rho A). \tag{4.6}$$

We see that ρ carries all the statistical information of a state in an ensemble, and if it is known, the average expectation value of any operator can be calculated with Eq. (4.6). Since $\mathrm{tr}(U^\dagger \rho A U) = \mathrm{tr}(\rho A)$, where U is a unitary transformation, the average value $\overline{\langle A\rangle}$ is independent of the choice of bases, as should be.

The time evolution of the density matrix can be obtained from the fundamental equation of quantum mechanics.

$$i\hbar \sum_n \frac{dc_n}{dt}|n\rangle = \sum_n c_n H|n\rangle. \tag{4.7}$$

Multiplying both sides of Eq. (4.7) by $\langle m|$, one obtains

$$i\hbar \frac{dc_m}{dt} = \sum_n c_n H_{mn}. \tag{4.8}$$

From Eq. (4.5),

$$\begin{aligned} \frac{d\rho_{nm}}{dt} &= \overline{c_n \frac{dc_m^*}{dt}} + \overline{c_m^* \frac{dc_n}{dt}} \\ &= \frac{i}{\hbar} \sum_k (\rho_{nk} H_{km} - H_{nk}\rho_{km}) \end{aligned} \tag{4.9}$$

or

$$\frac{d\rho}{dt} = \frac{i}{\hbar}[\rho, H]. \tag{4.10}$$

Exercise 4.1. Show that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, and therefore $\mathrm{tr}(U^\dagger \rho A U) = \mathrm{tr}(\rho A)$ when U is a unitary transformation.

4.2 Atomic Susceptibility

We shall consider the interaction of an ensemble of two-level atoms with the radiation field. For simplicity, we assume the radiation field is in a coherent state, and the average photon number is large such that it can be considered classically. The interaction Hamiltonian in the dipole approximation is

$$H_{\text{int}} = -e\mathbf{r} \cdot \mathbf{E}(t). \tag{4.11}$$

Let us make the direction of $\mathbf{E}$ the x-axis. Then in the two-level approximation

$$H_{\text{int}} = -ex\, E(t) = -\begin{bmatrix} \mu_{11} & \mu_{12} \\ \mu_{21} & \mu_{22} \end{bmatrix} E(t). \tag{4.12}$$

where $\mu_{mn} = \langle m|ex|n\rangle,\ m,n = 1,2$, and $|m\rangle, |n\rangle$ are the wavefunction of the lower and upper states respectively. For transitions between states of definite parity, $\mu_{11} = \mu_{22} = 0$. In addition, the phase of the wavefunction can be taken such that $\mu_{12} = \mu_{21} = \mu$. Substitute

$$\begin{aligned} H &= H_0 + H_{\text{int}} \\ &= \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} - \begin{bmatrix} 0 & \mu \\ \mu & 0 \end{bmatrix} E(t) \\ &= \begin{bmatrix} E_1 & -\mu E(t) \\ -\mu E(t) & E_2 \end{bmatrix} \end{aligned} \tag{4.13}$$

into Eq. (4.10), one obtains the equation of motion for the density matrix

$$\frac{d\rho_{21}}{dt} = -i\omega_0\rho_{21} + i\frac{\mu}{\hbar}E(t)\left(\rho_{11} - \rho_{22}\right), \tag{4.14}$$

$$\frac{d\rho_{22}}{dt} = -i\frac{\mu}{\hbar}E(t)\left(\rho_{21} - \rho_{21}^*\right), \tag{4.15}$$

where $\omega_0 = \left(E_2 - E_1\right)/\hbar$. The other two elements of the density matrix, ρ_{11} and ρ_{12}, can be decided from ρ_{22} and ρ_{21}, because $\rho_{11}+\rho_{22} = 1$ and $\rho_{12} = \rho_{21}^*$. Eq. (4.15) can be written as

$$\frac{d}{dt}\left(\rho_{11} - \rho_{22}\right) = 2i\frac{\mu}{\hbar}E(t)\left(\rho_{21} - \rho_{21}^*\right) \tag{4.16}$$

The average value of the dipole operator is given by

$$\overline{\langle\mu\rangle} = \sum_{m,n} \rho_{nm}\mu_{mn} = \mu\left(\rho_{12} + \rho_{21}\right). \tag{4.17}$$

Now we can see the interplay between the polarization $\overline{\langle\mu\rangle}$ and the population of the atomic level, $\rho_{11}-\rho_{22}$, under the influence of external field. Eq. (4.14) shows the population, together with $E(t)$, generates change of polarization. On the other hand, Eq. (4.16) shows the polarization and the external field generates change of population.

Population also changes due to spontaneous emission and external pumping. For simplicity, let us assume both the upper and the lower levels decay with the same time constant T_1. Including the effects of spontaneous emission and external pumping in Eq. (4.16), we have

$$\frac{d}{dt}\left(\rho_{11}-\rho_{22}\right)=2i\frac{\mu}{\hbar}E(t)\left(\rho_{21}-\rho_{21}^{*}\right)-\frac{\left(\rho_{11}-\rho_{22}\right)-\left(\rho_{11}-\rho_{22}\right)_0}{T_1}, \tag{4.18}$$

In Eq. (4.18) $\left(\rho_{11}-\rho_{22}\right)_0$ represents the steady state population difference when there is no stimulated emission. The steady state population difference $\left(\rho_{11}-\rho_{22}\right)_0$ is a function of temperature (thermal excitation) and external pumping. In the optical region, at room temperature the contribution of thermal excitation to $\left(\rho_{11}-\rho_{22}\right)_0$ is very small because of the large Boltzmann factor $\exp\left(-\hbar\omega/kT\right)$.

The off-diagonal matrix element ρ_{21} also has a decay constant T_2. The decay is due to the population decay as well as the phase randomization from the small differences in energy levels of the atoms in the ensemble. Including this effect in Eq. (4.14), we have

$$\frac{d\rho_{21}}{dt}=-i\omega_0\rho_{21}+i\frac{\mu}{\hbar}E(t)\left(\rho_{11}-\rho_{22}\right)-\frac{\rho_{21}}{T_2}. \tag{4.19}$$

Assuming $E(t)$ has the form $E_0\left[\exp(i\omega t)+\exp(-i\omega t)\right]/2$, and ρ_{21} has the same form $\sigma_{21}\exp(-i\omega t)+\sigma_{21}'\exp(i\omega t)$ since it is "driven" by $E(t)$, we can solve Eq. (4.19) immediately

$$\sigma_{21}=\frac{\mu E_0}{2\hbar}\frac{i\left(\rho_{11}-\rho_{22}\right)}{i\left(\omega_0-\omega\right)+1/T_2}, \tag{4.20}$$

$$\sigma_{21}'=\frac{\mu E_0}{2\hbar}\frac{i\left(\rho_{11}-\rho_{22}\right)}{i\left(\omega_0+\omega\right)+1/T_2}. \tag{4.21}$$

Since we are only interested in the near resonance case, *i.e.*, $\omega\approx\omega_0$, we can omit σ_{21}' because it is much smaller than σ_{21}. Near steady state, $d\left(\rho_{11}-\rho_{22}\right)/dt\approx 0$, Eq. (4.18) becomes

$$\left(\rho_{11}-\rho_{22}\right)\approx 2i\frac{\mu}{\hbar}E(t)T_1\left(\rho_{21}-\rho_{21}^{*}\right)+\left(\rho_{11}-\rho_{22}\right)_0, \tag{4.22}$$

then (4.20) implies

$$\left[i(\omega_0 - \omega) + \frac{1}{T_2}\right]\sigma_{21} = -2\Omega^2 T_1(\sigma_{21} - \sigma_{21}^*) + i\Omega(\rho_{11} - \rho_{22})_0, \tag{4.23}$$

where $\Omega = \mu E_0/2\hbar$. Because we are only interested in the slow varying time-average of ρ, we have omitted the high frequency components. From the real part and imaginary part of Eq. (4.23) one obtains

$$-(\omega_0 - \omega)\mathrm{Im}\sigma_{21} + \frac{1}{T_2}\mathrm{Re}\sigma_{21} = 0 \tag{4.24}$$

$$(\omega_0 - \omega)\mathrm{Re}\sigma_{21} + \frac{1}{T_2}\mathrm{Im}\sigma_{21} = -4\Omega^2 T_1 \mathrm{Im}\sigma_{21} + \Omega(\rho_{11} - \rho_{22})_0. \tag{4.25}$$

Hence

$$\mathrm{Im}\sigma_{21} = \frac{\Omega T_2(\rho_{11} - \rho_{22})_0}{1 + (\omega_0 - \omega)^2 T_2^2 + 4\Omega^2 T_2 T_1}, \tag{4.26}$$

$$\mathrm{Re}\sigma_{21} = \frac{(\omega_0 - \omega) T_2^2 \Omega(\rho_{11} - \rho_{22})_0}{1 + (\omega_0 - \omega)^2 T_2^2 + 4\Omega^2 T_2 T_1}. \tag{4.27}$$

With Eqs. (4.20), (4.26), and (4.27) one has

$$(\rho_{11} - \rho_{22}) = \frac{1 + (\omega_0 - \omega)^2 T_2^2}{1 + (\omega_0 - \omega)^2 T_2^2 + 4\Omega^2 T_2 T_1}(\rho_{11} - \rho_{22})_0, . \tag{4.28}$$

Substituting σ_{21} into Eq. (4.17), the polarization density $P(t)$ is

$$\begin{aligned} P(t) &= \mu N(\rho_{12} + \rho_{21}) = \mu N\left[\sigma_{21}\exp(-i\omega t) + \sigma_{21}^*\exp(i\omega t)\right] \\ &= 2\mu N|\sigma_{21}|\cos(\phi - \omega t), \end{aligned} \tag{4.29}$$

where N is the density of the two-level atoms, and $\phi = \arg(\sigma_{21})$. In the phasor notation $\tilde{E}(t) = \tilde{E}_0\exp(-i\omega t)$. From $\tilde{P}(t) = \chi\tilde{E}(t)$ and Eq. (4.29) where $\tilde{P}(t) = 2\mu N\sigma_{21}\exp(-i\omega t)$, the susceptibility χ can be obtained

$$\chi = \frac{2N\mu\sigma_{21}}{\tilde{E}_0}. \tag{4.30}$$

4.3 Amplification of Light

In the last paragraph we have derived the relation between the total polarization $P(t)$ and the population difference $\rho_{11} - \rho_{22}$. Now we shall see how the radiation field is changed by the polarization. From the one dimensional Maxwell equation

$$\frac{\partial^2 \tilde{E}}{\partial z^2} - \frac{\partial^2 \tilde{E}}{c^2 \partial t^2} = \frac{4\pi \partial^2 \tilde{P}}{c^2 \partial t^2}, \tag{4.31}$$

one can see that the electric field $\tilde{E}(t)$ can be driven by an oscillating polarization $\tilde{P}(t)$. Substituting $\tilde{P}(t) = \chi \tilde{E}(t)$ into Eq.(4.31), and solving for the slow-amplitude-variation solution $\tilde{E}(z,t) = \tilde{E}_0(z) \exp[i(kz - \omega t)]$, one finds

$$\frac{\partial \tilde{E}_0(z)}{\partial z} = \frac{2\pi i \omega}{c} \chi \tilde{E}_0(z), \tag{4.32}$$

or $\tilde{E}_0(z) = \tilde{E}_0(0) \exp(2\pi i \omega \chi z / c)$. We see that the radiation field is amplified when $\mathrm{Im}\chi$ is negative. From Eqs. (4.30), (4.26) and (4.28) we see amplification occurs when the population difference $\rho_{11} - \rho_{22}$ is negative (inverted). In thermal equilibrium, the population is determined by the Boltzmann distribution, $\rho_{11} - \rho_{22}$ is positive and the radiation will be attenuated (absorbed). Negative population difference can only be obtained with external pumping (excitation).

4.4 Laser Rate Equations

With a change of variable $z = ct$ and Eq. (4.32), one can derive the equation of motion for the field intensity $I(t) = cE_0^2/8\pi$, where E_0 is the amplitude of the electric field.

$$\frac{dI(t)}{dt} = -4\pi\omega \mathrm{Im}\chi I(t) - \frac{I(t)}{T_\mathrm{c}}. \tag{4.33}$$

In Eq (4.33) T_c is a phenomenological constant representing the photon loss rate. The term $-4\pi\omega\mathrm{Im}\chi$ represents the power gain. Since for coherent states with large photon number, $I(t)$ is equivalent to the photon flux, Eq. (4.33) can also be written as

$$\frac{dn}{dt} = -4\pi\omega \mathrm{Im}\chi n - \frac{n}{T_\mathrm{c}}, \tag{4.34}$$

where n is the photon density.

One can also derive a similar equation from Eq. (4.18) for the population inversion $\Delta N \equiv N(\rho_{22} - \rho_{11})$. Assuming $E(t)$ has the form $E_0[\exp(i\omega t) + \exp(-i\omega t)]/2$, we have

$$2i\frac{\mu}{\hbar} E(t) \left(\rho_{21} - \rho_{21}^*\right) = 2i \frac{\mu}{\hbar} \frac{E_0}{2} (2i \mathrm{Im}\sigma_{21}), \tag{4.35}$$

where we have ignored the irrelevant high frequency terms as we did in deriving Eq. (4.23). From Eq. (4.30) we have

$$2i\frac{\mu}{\hbar}E(t)\left(\rho_{21}-\rho_{21}^{*}\right)=-\frac{E_0^2}{N\hbar\omega}\omega\mathrm{Im}\chi. \tag{4.36}$$

Since $E_0^2/8\pi$ is the energy density, $E_0^2/8\pi\hbar\omega$ is the photon density n,

$$2i\frac{\mu}{\hbar}E(t)\left(\rho_{21}-\rho_{21}^{*}\right)=-\frac{8\pi\omega\mathrm{Im}\chi n}{N}. \tag{4.37}$$

Substitute Eq. (4.37) into Eq. (4.18), we obtain

$$\frac{d\Delta N}{dt}=8\pi\omega\mathrm{Im}\chi n-\frac{\Delta N}{T_1}+\frac{\Delta N_0}{T_1}, \tag{4.38}$$

where $\Delta N_0 \equiv N(\rho_{22}-\rho_{11})_0$. From Eqs. (4.30) and (4.26), (4.28),

$$\mathrm{Im}\chi=\frac{2N\mu}{E_0}\mathrm{Im}\sigma_{21}=-\frac{\mu^2 T_2\Delta N}{\hbar\left[1+(\omega_0-\omega)^2T_2^2\right]}, \tag{4.39}$$

$\mathrm{Im}\chi$ is proportional to ΔN. Therefore Eqs. (4.34), (4.38) can be written as

$$\frac{dn}{dt}=c\sigma\Delta Nn-\frac{n}{T_{\mathrm{c}}}, \tag{4.40}$$

$$\frac{d\Delta N}{dt}=-2c\sigma\Delta Nn-\frac{\Delta N}{T_1}+\frac{\Delta N_0}{T_1}, \tag{4.41}$$

where

$$\sigma=\frac{4\pi\mu^2}{c\hbar}\frac{\omega T_2}{1+(\omega_0-\omega)^2T_2^2}, \tag{4.42}$$

is the stimulated emission cross section. Eqs. (4.40) and (4.41) are referred as laser rate equations.

The cross section is frequency dependent, and greatly enhanced at the resonance frequency ω_0. The integrated cross section is

$$\int\sigma(\omega)\,\frac{d\omega}{\omega_0}=\frac{4\pi\mu^2}{c\hbar}\int\frac{\omega T_2}{1+(\omega-\omega_0)^2T_2^2}\,\frac{d\omega}{\omega_0}. \tag{4.43}$$

Change variable to $\nu=(\omega-\omega_0)T_2$, Eq. (4.43) becomes

$$\begin{aligned}\int\sigma\,\frac{d\omega}{\omega_0} &= \frac{4\pi\mu^2}{c\hbar}\int\frac{\nu+\omega_0T_2}{1+\nu^2}\,\frac{d\nu}{\omega_0T_2}\\ &= \frac{4\pi\mu^2}{c\hbar}\left(\frac{1}{\omega_0T_2}\int\frac{\nu d\nu}{1+\nu^2}+\int\frac{d\nu}{1+\nu^2}\right)\\ &= \frac{4\pi^2\mu^2}{c\hbar}.\end{aligned} \tag{4.44}$$

Since $\mu = e\langle 1|x|2\rangle$,

$$\frac{4\pi^2\mu^2}{c\hbar} = \frac{4\pi^2 e^2}{c\hbar}|\langle 1|x|2\rangle|^2 \approx \frac{4\pi^2}{137}|\langle 1|x|2\rangle|^2, \tag{4.45}$$

where $e^2/(c\hbar) \approx 1/137$ is the fine structure constant. Eq. (4.45) shows that the integrated cross section is of the same order as the size of the atom. However, for a sharp resonance the cross section at resonance frequency can be orders of magnitude larger than the size of the atom.

Chapter 5

Optical Resonators

5.1 Paraxial Approximation and Gaussian Beams

Optical resonator is an important part of lasers. In chapter 1 we show by an example that optical resonators lower the lasing threshold, so that the pump power can be reduced to a managable level. In this chapter we shall see that optical resonators are also responsible for the mode structures of lasers. Optical resonators are usually made of two parallel mirrors separated by a distance much larger than the wavelength. The separation is required to accommodate the gain medium and the intracavity optics. Due to the finite size of the mirrors, lightwave is required to propagate along the axis formed by the mirrors, with only a small angular spread. The spread is compensated by focusing elements in the cavity, such as intracavity lenses or concave cavity mirrors. We desinate the axis connecting the cavity mirrors the z-axis of our coordinate system. Since the projection of the wavevector is mainly on the z-axis, we can factor out the fast component along the z-axis from the solution of the wave equation and concentrate on the slow varying amplitude.

$$A(x, y, z) = u(x, y, z)e^{ikz}, \tag{5.1}$$

where $A(x, y, z)$ is the vector potential which is transverse to the z-axis. Substitute Eq. (5.1) into the wave equation $\nabla^2 A + k^2 A = 0$ and neglect $\partial^2 u/\partial z^2$ compared with $k\partial u/\partial z$, we obtain the paraxial wave equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 2ik\frac{\partial u}{\partial z} = 0. \tag{5.2}$$

In addition to the paraxial approximation, the solutions of Eq. (5.2) we are looking for must also fall off rapidly with increasing x and y, otherwise the light beam would not be confined in the optical resonator. Assume the mirrors have uniform reflectance, then the curvatures and the separation distance of the mirrors determine the boundary conditions. Let us consider first a most common optical resonator made of spherical mirrors. A particular solution of the wave equation $\nabla^2 A + k^2 A = 0$ which has a spherical phase front is the paraxial approximation of a spherical wave.

$$\begin{aligned} r &\approx z + \frac{x^2+y^2}{2z}, \\ g(x,y,z) &= \frac{e^{ikr}}{r} \approx \frac{e^{ikz}}{z} \exp\left(ik\frac{x^2+y^2}{2z}\right). \end{aligned} \tag{5.3}$$

But this solution is not a desirable solution for the resonator because it does not fall off rapidly with increasing x and y. Moreover, it requires that the distance between the mirrors to be exactly the sum of their radius curvatures. This is a impractical constrain in reality. A good solution can be obtained by noting that Eq. (5.2) has translational invariance on the z-axis. Therefore

$$g(x,y,z) = \frac{e^{ik(z+c)}}{(z+c)} \exp\left[ik\frac{x^2+y^2}{2(z+c)}\right]. \tag{5.4}$$

is also a solution of Eq. (5.2) for any complex number c [2]. As we shall see, any c with a nonzero imaginary part not only makes the solution fall off rapidly with increasing x and y, but also removes the constrain on the mirror separation. The real part and the imaginary part can be adjusted to make the phase front of the solution fit both mirrors simultaneously. Let us choose the origin such that at that point the phase front is flat. Then the solution is

$$g(x,y,z) = \frac{e^{ik(z-ib)}}{(z-ib)} \exp\left[ik\frac{x^2+y^2}{2(z-ib)}\right]. \tag{5.5}$$

After normalization and separation of real and imaginary exponents,

$$u(x,y,z) = \sqrt{\frac{2}{\pi}}\,\frac{e^{-i\phi}}{w} \exp\left(-\frac{x^2+y^2}{w^2}\right) \exp\left[\frac{ik}{2R}\left(x^2+y^2\right)\right], \tag{5.6}$$

where

$$b = \frac{\pi w_0^2}{\lambda}, \tag{5.7}$$

$$w^2(z) = \frac{2b}{k}\left(1+\frac{z^2}{b^2}\right), \tag{5.8}$$

$$\begin{aligned} \frac{1}{R(z)} &= \frac{z}{z^2+b^2}, &(5.9)\\ \phi(z) &= \arctan\left(\frac{z}{b}\right). &(5.10)\end{aligned}$$

Eq. (5.6) is a particular solution of the paraxial wave equation. Now we search for the general solutions. To make the task simpler, we first assume the general solution is of the following form:

$$u(x,y,z) = u_x(x,z)u_y(y,z). \tag{5.11}$$

Then $u_x(x,z)$ satisfies the two-dimensional paraxial equation

$$\frac{\partial^2 u}{\partial x^2} + 2ik\frac{\partial u}{\partial z} = 0, \tag{5.12}$$

and similarly for $u_y(y,z)$. Eq. (5.6) gives us a hint of the possible forms of the general solutions. Let us assume the general form to be

$$u_x(x,z) = \frac{e^{-i\theta(z)}}{\sqrt{w}} f\left(\frac{\sqrt{2}x}{w}\right)\exp\left(\frac{ikx^2}{2R}\right), \tag{5.13}$$

where w and R changes with z according to Eqs. (5.8) and (5.9), and θ and f are unknown functions to be solved. Substitute Eq. (5.13) into Eq. (5.12), we obtain

$$\frac{d^2 f}{d\xi^2} + \left[\frac{2(z^2+b^2)}{b}\frac{d\theta}{dz} - \xi^2\right] f = 0, \tag{5.14}$$

where $\xi = \sqrt{2}x/w$. Eq. (5.14) is the Schrödinger equation for the quantum harmonic oscillators. To have rapid fall-off solutions we must require that

$$\frac{2(z^2+b^2)}{b}\frac{d\theta}{dz} = 2n+1, \tag{5.15}$$

where $n = 0, 1, 2, \ldots$. For each n, the solution f is a Hermite-Gaussian function of order n. Since Hermite-Gaussian functions are orthogonal to each other and form a complete set, we may write any solution as a unique expansion of Hermite-Gaussian functions.

$$u_x(x,z) = \sum_n c_n \frac{e^{-i\theta(z)}}{\sqrt{w}} H_n\left(\frac{\sqrt{2}x}{w}\right)\exp\left(-\frac{x^2}{w^2}\right)\exp\left(\frac{ikx^2}{2R}\right). \tag{5.16}$$

The phase function $\theta(z)$ can be integrated from Eq. (5.15) to be

$$\theta(z) = \frac{2n+1}{2}\arctan\left(\frac{z}{b}\right). \tag{5.17}$$

Solutions for three dimensional cases can be obtained by combining two dimensional solutions.

$$\begin{aligned} u_{mn}(x,y,z) &= c_{mn}\frac{e^{-i(m+n+1)\phi}}{w}\Psi_m\left(\frac{\sqrt{2}x}{w}\right)\Psi_n\left(\frac{\sqrt{2}y}{w}\right) \\ & \exp\left[\frac{ik}{2R}\left(x^2+y^2\right)\right], \end{aligned} \tag{5.18}$$

where Ψ_i is the ith order Hermite-Gaussian function, and again

$$b = \frac{\pi w_0^2}{\lambda}, \tag{5.19}$$

$$w^2(z) = \frac{2b}{k}\left(1+\frac{z^2}{b^2}\right), \tag{5.20}$$

$$\frac{1}{R(z)} = \frac{z}{z^2+b^2}, \tag{5.21}$$

$$\phi(z) = \arctan\left(\frac{z}{b}\right). \tag{5.22}$$

These solutions can be used in series expansion to match any boundary condition.

The beam radius of a Gaussian mode is $w(z)$, and its minimum value is $w_0 = \sqrt{2b/k}$. The position of minimum beam radius is called the beam waist. The parameter $b = \pi w_0^2/\lambda$ is called the confocal parameter. At the distance b from the beam waist the beam radius becomes twice. At a large distance from the beam waist, $w \propto z$, and the diffraction angle θ satisfies

$$\tan(\theta) = \frac{w}{z} \approx \theta = \frac{\lambda}{\pi w_0}. \tag{5.23}$$

5.2 Optical Resonators with Curved Mirrors

The phase front of a Hermite-Gaussian mode is curved, with the inverse radius $1/R(z) = z/(z^2+b^2)$. One can choose suitable mirrors whose curvatures match the curved phase front to resonate with a Hermite-Gaussian mode. Conversely, for a fixed set of mirrors, there may exist modes which resonate with the cavity. For a set of mirror with curvature R_1, R_2 and separation d, the resonance conditions are

$$z_1 + z_2 = d, \tag{5.24}$$

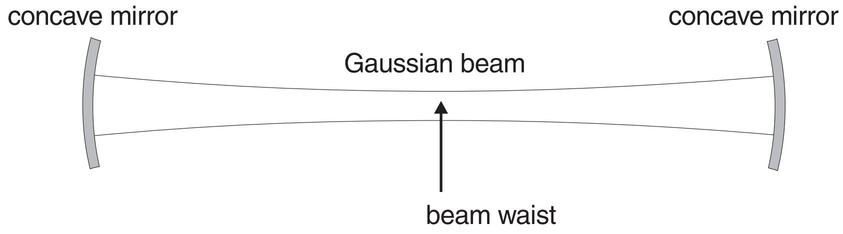

Figure 5.1: An optical resonator formed by two concave mirrors.

$$\frac{1}{R_1} = \frac{z_1}{z_1^2 + b^2}, \tag{5.25}$$

$$\frac{1}{R_2} = \frac{z_2}{z_2^2 + b^2}. \tag{5.26}$$

As an example, let us consider one particular case where the mirror curvatures $R_1 = R_2 = R$ and the separation between the mirrors is d. Setting $R(d/2) = R$ in Eq. (5.9) and solve for b, we obtain

$$b = \sqrt{\frac{d}{2}\left(R - \frac{d}{2}\right)}. \tag{5.27}$$

In the general case, one can solve b^2 from Eqs. (5.24)-(5.26)

$$4b^2 = R_1^2 R_2^2 \frac{1 - \left[\frac{2d(d-R_1-R_2)}{R_1 R_2} + 1\right]^2}{(2d - R_1 - R_2)^2}. \tag{5.28}$$

For a stable mode, b must be real. Some manipulation leads to the following stability condition.

$$0 \leq \left(1 - \frac{d}{R_1}\right)\left(1 - \frac{d}{R_2}\right) \leq 1. \tag{5.29}$$

Boundary conditions of the resonator also limit the choice of k. For the symmetric cavity considered above, the periodic boundary condition requires that

$$kd - 2(m + n + 1)\arctan\left(\frac{d}{2b}\right) + \theta_0 = l\pi, \tag{5.30}$$

where l is a positive integer which labels the longitudinal modes, and θ_0 is the phase shift produced by the mirrors. In the Gaussian beam solutions for the optical resonators we should replace k by k_l to indicate the restriction on k.

5.3 Transverse Degrees of Freedom

In section 5.1 we found out the complete set of Gaussion-beam solutions of the paraxial equation by somewhat educated guess. Even though for most applications knowing the solutions is already good enough, intellectually it is rewarding if one could gain some deeper insight of the solutions. In this section we shall derive these solutions from physical principles, in such a way that the origin and meaning of the transverse modes will become clear.

In geometric optics, if an optical ray propagates nearly along the z-axis, *i.e.*, the angle between the ray and the z-axis is small, the trajectory of the ray is described by the following equation.

$$\begin{bmatrix} r_{\rm out} \\ r'_{\rm out} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} r_{\rm in} \\ r'_{\rm in} \end{bmatrix}, \tag{5.31}$$

where r is the position of the ray, r' is the slope of the ray, and the transformation matrix is called the ABCD matrix. For free propagation,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}, \tag{5.32}$$

where z is the distance of propagation. Whereas for a ray passing through a lens,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}. \tag{5.33}$$

Consider an optical cavity made of a flat mirror at $z = 0$ and a concave mirror at $z = d$. An optical ray starting from the flat mirror described by (r_n, r'_n) at the nth round trip returns to the flat mirror with a new position and a new slope (r_{n+1}, r'_{n+1}). The position and slope are related by

$$\begin{bmatrix} r_{n+1} \\ r'_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r_n \\ r'_n \end{bmatrix}, \tag{5.34}$$

where R is the radius of curvature of the concave mirror. We may change notation from r to x, and note that $r' = p_x/p_z \approx p_x/(\hbar k)$, then Eq. (5.34) becomes

$$\begin{bmatrix} x_{n+1} \\ p_{n+1} \end{bmatrix} = \begin{bmatrix} 1 - 2d/R & [2d/(\hbar k)](1 - d/R) \\ -2\hbar k/R & 1 - 2d/R \end{bmatrix} \begin{bmatrix} x_n \\ p_n \end{bmatrix}, \tag{5.35}$$

or

$$\begin{bmatrix} x_{n-1} \\ p_{n-1} \end{bmatrix} = \begin{bmatrix} 1 - 2d/R & [2d/(\hbar k)](1 - d/R) \\ -2\hbar k/R & 1 - 2d/R \end{bmatrix}^{-1} \begin{bmatrix} x_n \\ p_n \end{bmatrix}. \tag{5.36}$$

That implies

$$\begin{aligned}\frac{x_{n+1}-x_{n-1}}{2} &= \frac{2d}{\hbar k}\left(1-\frac{d}{R}\right)p_n, &(5.37)\\ \frac{p_{n+1}-p_{n-1}}{2} &= -\frac{2\hbar k}{R}x_n. &(5.38)\end{aligned}$$

The equation of motion that corresponds to Eqs. (5.37) and (5.38) is

$$\begin{aligned}\frac{\partial H}{\partial p} &= \frac{dx}{dt}=\frac{2d}{\hbar k}\left(1-\frac{d}{R}\right)p,\\ \frac{\partial H}{\partial x} &= -\frac{dp}{dt}=\left(\frac{2\hbar k}{R}\right)x, \qquad (5.39)\end{aligned}$$

where the Hamiltonian function is

$$H=\frac{d}{\hbar k}\left(1-\frac{d}{R}\right)p^2+\left(\frac{\hbar k}{R}\right)x^2. \qquad (5.40)$$

In wave mechanics, p is replaced by $-i\hbar d/dx$. If the wave in the optical cavity satisfies the resonance condition, the waveform remains stationary. Therefore the wave at the flat mirror, $z=0$, must be

$$\psi_n(x,0)=c_nH_n\left(\sqrt{\frac{k}{b}}x\right)\exp\left(-\frac{1}{2}\frac{k}{b}x^2\right), \qquad (5.41)$$

where $b=\sqrt{dR-d^2}$, $c_n=2^{-n/2}\pi^{-1/4}(n!)^{-1/2}(k/b)^{1/4}$ is the normalization constant, and $H_n(x)$ is the Hermite function of degree n. This solution can serve as the initial condition to find the solution, $\psi_n(x,z)$, of the paraxial equation Eq. (5.2) for any z.

Let us express $\psi_n(x,z)$ in terms of its Fourier components.

$$\psi_n(x,z)=\frac{1}{\sqrt{2\pi}}\int\phi(k_x,z)e^{ik_xx}\,dk_x. \qquad (5.42)$$

Substitute it into Eq. (5.2), one has

$$\frac{\partial\phi(k_x,z)}{\partial z}=\frac{-ik_x^2}{2k}\phi(k_x,z). \qquad (5.43)$$

Therefore

$$\phi(k_x,z)=\phi(k_x,0)\exp\left(-\frac{ik_x^2z}{2k}\right). \qquad (5.44)$$

Since

$$\phi(k_x, 0) = \frac{1}{\sqrt{2\pi}} \int \psi_n(x', 0) e^{-ik_x x'} \, dx', \tag{5.45}$$

one has

$$\psi_n(x, z) = \frac{1}{2\pi} \iint \psi_n(x', 0) \exp\left[ik_x(x - x') - \frac{ik_x^2 z}{2k}\right] dk_x dx'. \tag{5.46}$$

Using the formula

$$\int \exp(-ax^2) \, dx = \sqrt{\frac{\pi}{a}}, \tag{5.47}$$

Eq. (5.46) becomes

$$\psi_n(x, z) = \sqrt{\frac{k}{i2\pi z}} \int \psi_n(x', 0) \exp\left[\frac{ik}{2z}(x - x')^2\right] dx'. \tag{5.48}$$

Substitute Eq. (5.41) into Eq. (5.48) and change the variable $\sqrt{k/b}x'$ to x', Eq. (5.48) becomes

$$\psi_n\left(\sqrt{\frac{b}{k}}x, z\right) = c_n \sqrt{\frac{b}{i2\pi z}} \int H_n(x') \exp\left(-\frac{1}{2}x'^2\right) \exp\left[\frac{ib}{2z}(x - x')^2\right] dx' \tag{5.49}$$

Set $a = -ib/z$, the integration is $\int H_n(x') e^{-x'^2/2} e^{-a(x-x')^2/2} dx'$. We know that the generating function of Hermite polynomials is $\exp\left(-s^2 + 2sx'\right)$ [1], i.e.,

$$\exp\left(-s^2 + 2sx'\right) = \sum_{n=0}^{\infty} \frac{s^n}{n!} H_n(x'). \tag{5.50}$$

Therefore

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{s^n}{n!} \int H_n(x') e^{-x'^2/2} e^{-a(x-x')^2/2} dx' \\
= & \int \exp\left(-s^2 + 2sx'\right) e^{-x'^2/2} e^{-a(x-x')^2/2} dx' \\
= & \sqrt{\frac{2\pi}{a+1}} \exp\left(-\frac{a-1}{a+1}s^2 + \frac{2a}{a+1}sx\right) \exp\left[-\frac{ax^2}{2(a+1)}\right].
\end{aligned} \tag{5.51}$$

Again by Eq. (5.50),

$$\exp\left(-\frac{a-1}{a+1}s^2 + \frac{2a}{a+1}sx\right) = \sum_{n=0}^{\infty} \frac{s^n}{n!} \left(\frac{a-1}{a+1}\right)^{n/2} H_n\left(\frac{ax}{\sqrt{a^2-1}}\right). \tag{5.52}$$

Substitute Eq. (5.50) into Eq. (5.51) and compare both sides of Eq. (5.51), one obtains

$$\begin{aligned} & \int H_n(x')e^{-x'^2/2}e^{-a(x-x')^2/2}dx' \\ = & \sqrt{\frac{2\pi}{a+1}}\left(\frac{a-1}{a+1}\right)^{n/2} H_n\left(\frac{ax}{\sqrt{a^2-1}}\right)\exp\left[-\frac{ax^2}{2(a+1)}\right]. \end{aligned} \tag{5.53}$$

From Eqs. (5.49) and (5.53),

$$\begin{aligned} \psi_n(x,z) &= 2^{-n/2}\pi^{-1/4}(n!)^{-1/2}\,(kb)^{1/4}\sqrt{\frac{1}{b+iz}}\left(\frac{b-iz}{b+iz}\right)^{n/2} \\ & \qquad H_n\left(\sqrt{\frac{bk}{z^2+b^2}}x\right)\exp\left(-\frac{k}{2}\frac{b-iz}{z^2+b^2}x^2\right) \\ &= 2^{-n/2+1/4}\pi^{-1/4}(n!)^{-1/2}\frac{e^{-i\theta(z)}}{\sqrt{w}} \\ & \qquad H_n\left(\frac{\sqrt{2}x}{w}\right)\exp\left(-\frac{x^2}{w^2}\right)\exp\left(\frac{ikx^2}{2R}\right), \end{aligned} \tag{5.54}$$

where w, R, and $\theta(z)$ are defined in Eq. (5.20), (5.21), and (5.17) respectively.

5.4 Propagation of Gaussian Beams

A Hermite-Gaussian beam is completely described by the parameter $q \equiv z - ib$.

$$\frac{1}{q} = \frac{1}{R(z)} + \frac{i\lambda}{\pi w^2}. \tag{5.55}$$

To know how the Gaussian beam propagates we only need to know how the q parameter transforms.

case 1. free space of length d,

$$q' = z' - ib = z + d - ib = q + d. \tag{5.56}$$

case 2. a thin lens of focal length f. A thin lens advances the phase by $-ik(x^2+y^2)/2f$, therefore

$$\frac{1}{R'} = \frac{1}{R} - \frac{1}{f}, \tag{5.57}$$

$$\frac{1}{q'} = \frac{1}{q} - \frac{1}{f}. \tag{5.58}$$

In general the transformation of the q-parameter can be written in the following form

$$q' = \frac{Aq + B}{Cq + D}, \tag{5.59}$$

where A, B, C, D are the matrix elements of the so called $ABCD$ matrix. The form in Eq. (5.59) is convenient for computation because cascading transforms can be combined into a single transform by multiplying the $ABCD$ matrices together. One can easily verify that if

$$q_1 = \frac{A_0 q_0 + B_0}{C_0 q_0 + D_0}, \quad q_2 = \frac{A_1 q_1 + B_1}{C_1 q_1 + D_1},$$

then

$$q_2 = \frac{A q_0 + B}{C q_0 + D},$$

where

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} \tag{5.60}$$

For the two example cases above, the $ABCD$ matrices are

$$\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$$

respectively.

To show the usefulness of $ABCD$ matrices, let us consider the problem of focusing a Gaussian beam with a lens of focal length be f. At a distance d from the lens the combined $ABCD$ matrix is

$$\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} = \begin{bmatrix} 1 - \frac{d}{f} & d \\ -\frac{1}{f} & 1 \end{bmatrix}. \tag{5.61}$$

Let $q(0)$ and $q(d)$ be the q parameters at the lens and at distance d respectively. With the $ABCD$ matrix above, we find

$$\mathrm{Im}\left[\frac{1}{q(d)}\right] = \frac{\mathrm{Im}\,[1/q(0)]}{\{\mathrm{Re}\,[1/q(0)]\,d + 1 - d/f\}^2 + \{\mathrm{Im}\,[1/q(0)]\,d\}^2}. \tag{5.62}$$

The beam waist is located at d_0, where

$$\frac{d}{dz}\mathrm{Im}\left[\frac{1}{q(z)}\right]\Bigg|_{z=d_0} = 0. \tag{5.63}$$

From Eq. (5.63) one obtains

$$d_0 = \frac{1/f - \mathrm{Re}\,[1/q(0)]}{\{1/f - \mathrm{Re}\,[1/q(0)]\}^2 + \{\mathrm{Im}\,[1/q(0)]\}^2}. \tag{5.64}$$

Subsitute Eq. (5.64) into Eq. (5.62) we get

$$\mathrm{Im}\left[\frac{1}{q(d_0)}\right] = \mathrm{Im}\,[1/q(0)]\,(\alpha + 1), \tag{5.65}$$

where

$$\alpha \equiv \left\{\frac{\mathrm{Re}\,[1/q(0)] - 1/f}{\mathrm{Im}\,[1/q(0)]}\right\}^2.$$

If the incoming beam is close to a plane wave, then $\mathrm{Re}[1/q(0)] \to 0$ and $\mathrm{Im}[1/q(0)] \ll 1/f$. Eq. (5.65) becomes

$$\mathrm{Im}\left[\frac{1}{q(d_0)}\right] = \frac{1}{f^2 \mathrm{Im}\,[1/q(0)]}. \tag{5.66}$$

Because the size of the incoming beam must be smaller than the lens, $\mathrm{Im}[1/q(0)] > 4\lambda/(\pi D^2)$, where D is the diameter of the lens, we have

$$w(d_0) > \frac{2\lambda f}{\pi D}. \tag{5.67}$$

Since light emitted from a far away source resemble a plane wave to the focusing lens, Eq. (5.67) reveals that the resolution limit of a lens is proportioinal to $\lambda f/D$.

In contrast, if a Gaussian beam is already tightly focused when it reaches the lens, then $\alpha \ll 1$. In this case $w(d_0) \approx w(0)$. This is not difficult to see as one realizes that a small beam does not experience the curvature of a lens, therefore the lens has little effect on the beam.

Exercise 5.1. A Gaussian mirror has a graded reflectivity $R(r) = \exp(-r^2/2g)$. Show that the $ABCD$ matrix is

$$\begin{bmatrix} 1 & 0 \\ \frac{i}{kg} & 1 \end{bmatrix}.$$

Exercise 5.2. A graded-index waveguide of length l has an index of refraction $n(r) = n_0(1 - r^2/2h^2)$. Show that the $ABCD$ matrix is

$$\begin{bmatrix} \cos\left(\frac{l}{h}\right) & \frac{h}{n_0}\sin\left(\frac{l}{h}\right) \\ -\frac{n_0}{h}\sin\left(\frac{l}{h}\right) & \cos\left(\frac{l}{h}\right) \end{bmatrix}.$$

Exercise 5.3. How does a Gaussian beam propagate through the interface between two materials with different index of refraction? Assume the wavevector is normal to the interface.

Exercise 5.4. How does a Gaussian beam propagate through the interface between two materials with different index of refraction, if the wavevector is not normal to the interface?

5.5 Self-Consistent q-Parameter

In an optical resonator, the q-parameter of the resonant Gaussian mode must not change after a round trip in the resonator, i.e.

$$q = \frac{Aq+B}{Cq+D}, \tag{5.68}$$

where A, B, C, D is the matrix elements of the combined $ABCD$ matrix in a round trip. The q-parameter which satisfies Eq. (5.68) is called the self-consistent q-parameter. Eq. (5.68) can be used to find the q-parameter of the resonant Gaussian mode of a laser cavity.

$$q = \frac{(A-D) \pm \sqrt{(A-D)^2 + 4BC}}{2C}. \tag{5.69}$$

Since the q-parameter can not be a real number, $(A-D)^2 + 4BC$ must be negative. This is the existence condition of a resonant Gaussian mode.

Exercise 5.5. The condition for the existence of self-consistent q is $q = (Aq+B)/(Cq+D)$. As important as the existence of a solution is its stability. What is the stability condition of a self-consistent q? Hint: $f(q) \equiv (Aq + B)/(Cq + D)$, the stability condition is $|df(q)/dq| \leq 1$.

Solution: Since $AD - BC = 1$

$$\begin{aligned} df(q)/dq &= \frac{1}{(Cq+D)^2}, \\ |df(q)/dq| &= \frac{1}{|Cq+D|^2}. \end{aligned}$$

$$q = \frac{A - D \pm \sqrt{(A-D)^2 + 4BC}}{2C} = \frac{A - D \pm \sqrt{(A+D)^2 - 4}}{2C}.$$

$$Cq + D = \frac{A + D \pm \sqrt{(A + D)^2 - 4}}{2}.$$

Because the imaginary part of q cannot be zero, $(A + D)^2 - 4 < 0$.

$$|Cq + D|^2 = \frac{(A + D)^2 + [4 - (A + D)^2]}{4} = 1.$$

Therefore $|df(q)/dq| = 1$. A round-trip mapping of q near the self-consistent solution will not reduce the deviation, nor will amplify it. The deviation only change its phase in the complex plane.

Chapter 6

Quantum Theory of Lasers

6.1 Oscillation in a Laser Cavity

We have explained in chapter 4 that the interplay between the electric field and the polarization is governed by the following two equations:

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E} = \frac{4\pi}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{P}, \tag{6.1}$$

$$\mathbf{P} = \chi \mathbf{E}, \tag{6.2}$$

where χ can be expressed in terms of microscopic parameters and density matrix elements. In a laser cavity, we should describe the fields in terms of the laser cavity modes. Let us denote $\mathbf{E}_m(\mathbf{r})\exp(-i\omega_m t)$, $-i\mathbf{B}_m(\mathbf{r})\exp(-i\omega_m t)$ the field of the mth cavity mode. From Maxwell equation

$$\nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{c\partial t}, \tag{6.3}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{c\partial t}. \tag{6.4}$$

One can show that $\mathbf{E}_m(\mathbf{r})$, $\mathbf{B}_m(\mathbf{r})$ satisfy the following relation.

$$\nabla \times \mathbf{B}_m = \frac{\omega_m}{c}\mathbf{E}_m, \tag{6.5}$$

$$\nabla \times \mathbf{E}_m = \frac{\omega_m}{c}\mathbf{B}_m. \tag{6.6}$$

Expanding the field

$$\mathbf{E}(\mathbf{r},t) = \sum_m p_m(t)\mathbf{E}_m(\mathbf{r}), \tag{6.7}$$

$$\mathbf{B}(\mathbf{r},t) = \sum_m q_m(t)\mathbf{B}_m(\mathbf{r}). \tag{6.8}$$

and substituting into the Maxwell equation with the driving polarization

$$\nabla \times \mathbf{B} = \frac{\partial(\mathbf{E} + 4\pi\mathbf{P})}{c\partial t}, \tag{6.9}$$

we have

$$\sum_m q_m(t)\frac{\omega_m}{c}\mathbf{E}_m = \sum_m \frac{\dot{p}_m(t)}{c}\mathbf{E}_m + \frac{4\pi\partial\mathbf{P}}{c\partial t}, \tag{6.10}$$

or equivalently,

$$\sum_m \dot{q}_m(t)\omega_m\mathbf{E}_m = \sum_m \ddot{p}_m(t)\mathbf{E}_m + \frac{4\pi\partial^2\mathbf{P}}{\partial^2 t}, \tag{6.11}$$

From Eq. (6.4) and (6.6) we have

$$\sum_m p_m(t)\frac{\omega_m}{c}\mathbf{B}_m = \sum_m -\frac{\dot{q}_m(t)}{c}\mathbf{B}_m, \tag{6.12}$$

or

$$\dot{q}_m(t) = -\omega_m p_m(t). \tag{6.13}$$

Substituting this relation into Eq. (6.11) and multiplying both sides by $\mathbf{E}_n$, and integrating over the cavity volume, we have

$$-\omega_n{}^2 p_n(t) = \ddot{p}_n(t) + \frac{4\pi\partial^2}{\partial^2 t}\int_{\mathrm{c}} \mathbf{P}(\mathbf{r},t)\cdot\mathbf{E}_n(\mathbf{r})d^3r, \tag{6.14}$$

where the subscript c under the integration sign means that the integration is over the volume of the cavity. Eq. (6.14) does not include the loss in the cavity. We may emulate the loss by adding $-\eta\dot{p}_n(t)$ to the left hand side of Eq. (6.14). Replacing $\partial/\partial t$ by $-i\omega$ and substitute Eq. (6.2) in for $\mathbf{P}$, we have

$$\omega^2 - \omega_n{}^2 + i\omega\eta = -4\pi\omega^2 f(\mathrm{Re}\chi + i\mathrm{Im}\chi). \tag{6.15}$$

The f in Eq. (6.15) is called the filling factor. It is defined by

$$f = \int_{\mathrm{m}} \mathbf{E}_n(\mathbf{r})\cdot\mathbf{E}_n(\mathbf{r})d^3r, \tag{6.16}$$

where the subscript m means that the integration is over the laser medium. Since the laser medium does not necessarily fill out the whole cavity, $f \le 1$. As we shall see, the imaginary part of Eq. (6.15) determines the energy balance, hence the power of the laser, whereas the real part determines the laser frequency.

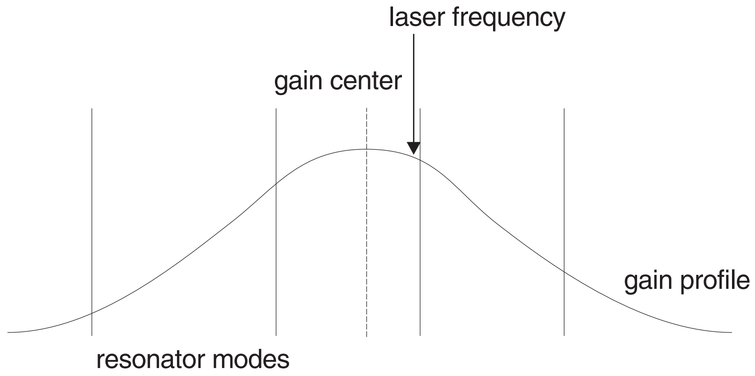

Figure 6.1: Frequency pulling between resonator modes and the gain center.

6.2 Laser Power and Frequency

In a lossy laser cavity the intensity decays according to $\exp(-t/T_c)$ when the population inversion is zero. While the term $i\omega\eta$ in Eq. (6.15) causes the field amplitude to decay according to $\exp(-\eta t/2)$ when $\chi = 0$. Therefore $\eta = 1/T_c$. From the imaginary part of Eq. (6.15), together with Eqs. (4.26), (4.28), and (4.30), we obtain the relation

$$\frac{1}{T_c} = -4\pi\omega f \mathrm{Im}\chi = \frac{-4\pi\omega f\mu^2}{\hbar}\frac{T_2 N(\rho_{11}-\rho_{22})}{1+(\omega_0-\omega)^2 T_2^2}. \tag{6.17}$$

Since $I(t) = cE_0^2/8\pi$, Eq. (6.17) determines the power of the laser. From Eq. (4.42) we can also write Eq. (6.17) in a more familiar form:

$$\frac{1}{T_c} = fc\sigma\Delta N, \tag{6.18}$$

which simply means gain equals to loss.

The frequency of the laser is determined by the real part of Eq. (6.15). From Eqs. (4.26) and (4.27),

$$\omega^2 - \omega_n^2 = -4\pi\omega^2 f \mathrm{Re}\chi = -4\pi\omega^2(\omega_0-\omega)T_2 f \mathrm{Im}\chi. \tag{6.19}$$

Since $\eta = -4\pi\omega f \mathrm{Im}\chi$ and $\omega^2 - \omega_n^2 \approx 2\omega(\omega-\omega_n)$, we have

$$\omega - \omega_n = \frac{\eta T_2}{2}(\omega_0 - \omega). \tag{6.20}$$

Usually $\eta T_2 \ll 1$, the laser frequency sits very close to the cavity resonance frequency, and is pulled slightly toward the gain center of the laser medium.

6.3 Laser Bandwidth

From the theory presented above it appears that a laser works at a single frequency, *i.e.* the spectrum is a δ-function. This is only because we omitted the spontaneous emission. Although Eqs. (6.1) and (6.2) are classical equations, we may emulate spontaneous emission by modifying Eq. (6.2) to

$$\mathbf{P} = \chi \mathbf{E} + \mathbf{P}_{\mathrm{sp}}, \tag{6.21}$$

where $\mathbf{P}_{\mathrm{sp}}$ represents the "spontaneous" polarization as opposed to the induced polarization which is proportional to $\mathbf{E}$. In the mean time we should assume that $p_n(t)$ in Eq. (6.14) has a spectrum $p_n(\omega)$. Then we may rewite Eq. (6.14) as

$$\begin{aligned} & \int (\omega^2 - \omega_n^2 + i\omega\eta + 4\pi\omega^2 f\chi) p_n(\omega) \exp(-i\omega t) d\omega \\ = & \int \omega^2 n_{\mathrm{sp}}(\omega) \exp(-i\omega t) d\omega, \end{aligned} \tag{6.22}$$

where

$$\int n_{\mathrm{sp}}(\omega) \exp(-i\omega t) d\omega = -4\pi \int_{\mathrm{c}} \mathbf{P}_{\mathrm{sp}}(\mathbf{r}, t) \cdot \mathbf{E}_n(\mathbf{r}) d^3 r. \tag{6.23}$$

The solution of Eq. (6.22) is

$$\begin{aligned} p_n(\omega) &= \frac{\omega^2 n_{\mathrm{sp}}(\omega)}{\omega^2 - \omega_n^2 + i\omega\eta + 4\pi\omega^2 f\chi} \\ &= \frac{\omega^2 n_{\mathrm{sp}}(\omega)}{G(\omega)} \frac{1}{[\omega^2 - \omega_n^2/G(\omega)] + i\omega\eta F(\omega)}, \end{aligned} \tag{6.24}$$

where

$$G(\omega) = 1 + 4\pi f \mathrm{Re}[\chi(\omega)], \quad F(\omega) = \frac{\eta + 4\pi\omega f \mathrm{Im}[\chi(\omega)]}{G(\omega)\eta}. \tag{6.25}$$

We expect that the spectrum of the laser is close to a δ-function. Therefore we may neglect the weak frequency dependence of the factor $\omega^2 n_{\mathrm{sp}}(\omega)$, $G(\omega)$, and $F(\omega)$. The shape of the spectrum is determined by the second factor in Eq. (6.24). The center frequency ω_{L} is determined by $\omega_{\mathrm{L}}^2 = \omega_n^2/(1+4\pi f \mathrm{Re}\chi)$ as expected. The bandwidth $\delta\omega$ is

$$\delta\omega = \frac{\eta}{2} F. \tag{6.26}$$

Without spontaneous emission $-4\pi\omega f \mathrm{Im}\chi = \eta$, hence $\delta\omega = 0$ as expected. With spontaneous emission η is slightly larger than $-4\pi\omega f \mathrm{Im}\chi$, but the difference is so small that the laser bandwidth is greatly reduced from the bandwidth of the passive cavity.

Now we show that the bandwidth reduction factor is related to the power of the laser. Noting that $\omega^2 - \omega_L^2 \approx 2\omega(\omega - \omega_L)$, we may simplify Eq. (6.24) to

$$p_n(\omega) = \frac{\omega n_{sp}(\omega)/[2G(\omega)]}{\omega - \omega_L + i\eta F(\omega)/2}. \tag{6.27}$$

If the laser oscillates in a single mode, then the energy in the laser cavity is

$$\begin{aligned} E_c &= \frac{1}{8\pi}\int\int_c \mathbf{E}(\mathbf{r},\omega)\cdot\mathbf{E}^*(\mathbf{r},\omega)d^3r d\omega \\ &= \frac{1}{8\pi}\int_{-\infty}^{\infty} |p_n(\omega)|^2 d\omega \\ &\approx \frac{1}{8\pi}\int_{-\infty}^{\infty} \frac{A}{(\omega-\omega_L)^2+\delta\omega^2}d\omega = \frac{A}{8\delta\omega}. \end{aligned} \tag{6.28}$$

where $A = \omega_L^2|n_{sp}(\omega_L)|^2/[4G(\omega_L)^2]$ and $\delta\omega = \eta(\omega_L)F(\omega_L)/2$. Therefore

$$\delta\omega = \frac{A}{8E_c} = \frac{A}{8n_c\hbar\omega}, \tag{6.29}$$

where n_c is the number of photons in the cavity. To go further, we can use our knowledge in the quantum theory of light to get rid of the quantity A. It is shown in section 2.5 that every mode of electromagnetic waves is equivalent to a harmonic oscillator, therefore for each single mode in the laser cavity, there is a zero-point energy $\hbar\omega/2$. In a passive cavity $\chi = 0$, $\delta\omega = \eta/2 \equiv \delta\omega_c$ and the energy is $\hbar\omega/2$. Therefore

$$A = 4\hbar\omega\delta\omega_c. \tag{6.30}$$

Put back in Eq. (6.29), we have

$$\delta\omega = \frac{\delta\omega_c}{2n_c}, \tag{6.31}$$

or equivalently

$$\delta\nu = \frac{\delta\nu_c}{2n_c}, \tag{6.32}$$

where $\delta\nu_c = \eta/(4\pi)$ is the bandwidth of the passive cavity. Eq. (6.32) shows that the ratio of laser frequency uncertainty $\delta\nu$ to the empty-cavity frequency uncertainty $\delta\nu_c$ is equal to the ratio of zero-point photon number 1/2 to the cavity photon number n_c. Because n_c is a large number, the bandwidth of the laser is greatly reduced by stimulated emission. Meanwhile, since spontaneous emission always comes together with stimulated emission, Eq. (6.32) represents the ultimate limit of laser bandwidth. Writing n_c in terms of the output power $P_{\text{out}} = \eta E_c = \eta n_c h\nu$, one has

$$n_c = \frac{P_{\text{out}}}{\eta h\nu} = \frac{P_{\text{out}}}{4\pi\delta\nu_c h\nu}. \tag{6.33}$$

Substitute Eq. (6.33) into Eq. (6.32), one has

$$\delta\nu = 2\pi\frac{h\nu\delta\nu_c^2}{P_{\text{out}}}. \tag{6.34}$$

An equation similar to Eq. (6.34) was first written down by Schawlow and Townes, and $\delta\nu$ is known as the Schawlow-Townes limit.

As an example, let us consider the bandwidth limit of a common 633-nm He-Ne laser. The cavity length is typically 30 cm, and the cavity loss is typically 1%. The cavity bandwidth $\delta\nu_c$ is $0.01 \times 3 \times 10^8/(4\pi \times 0.6) \approx 4 \times 10^6$ Hz. At 1-mW output power the number of photon in the cavity is $0.001/0.01/(h\nu c) \times 0.6 \approx 6.4 \times 10^8$. The bandwidth limit is $\approx 3 \times 10^{-3}$ Hz.

Eq. (6.34) gives the bandwith limit if spontaneous emission is the only source of noise. In reality, there are other factors that broaden the laser bandwidth. For example, consider Eq. (5.30) which determines the frequency of the cavity modes.

$$kd - 2(m+n+1)\arctan(d/2b) + \theta_0 = l\pi. \tag{6.35}$$

Density fluctuation of air causes a corresponding fluctuation of the optical path length d. We have $|\delta\nu/\nu| = |\delta k/k| = |\delta d/d| \approx \delta n$, where δn is the fluctuation of the index of refraction caused by density fluctuation. Since $n-1$ is proportional to the density ρ, we have

$$\begin{aligned} n &= 1 + c\rho \\ \delta n &= c\delta\rho = c\sqrt{N}/V = \frac{n-1}{\sqrt{N}}. \end{aligned} \tag{6.36}$$

where N is the number of air molecules in V, and we have used the thermodynamics relation $\delta N = \sqrt{N}$. For a 1-m laser cavity and 1-mm^2 beam cross section, at standard conditions the number of molecules covered by the beam is $\approx 2.7 \times 10^{19}$, and $n - 1 \approx 3 \times 10^{-4}$. Hence $\delta\nu/\nu \approx 6 \times 10^{-14}$. We see that the frequency fluctuation caused by thermodynamics density fluctuation can be much larger than the Schawlow-Townes limit.

In real life the bandwidth of a laser is usually limited by the cavity length fluctuation caused by vibrations from the environment. A 0.01-nm fluctuation (1/10 of the size of an atom) yields $\delta\nu/\nu \approx 10^{-11}$, which is much larger than both limits considered above.

Finally we note that from Eq. (6.26) and (6.31) we have

$$F = \frac{\delta\omega}{\delta\omega_c} = \frac{1}{2n_c}. \tag{6.37}$$

Eqs. (6.37) and (6.25) relate the output power to $\mathrm{Im}\chi$.

$$\frac{G}{2n_c} = 1 + \frac{4\pi\omega f \mathrm{Im}\chi}{\eta}. \tag{6.38}$$

Since $G/2n_c \ll 1$, $-4\pi\omega f \mathrm{Im}\chi \approx \eta$. This reduces to our previous result.

Chapter 7

Laser Dynamics

7.1 Relaxation Oscillation

In previous chapters we described the steady-state characteristics of lasers. Now we shall investigate the behavior of a laser when it is slightly away from the steady state. By performing a standard stability analysis on the laser rate equations, we see how the laser relaxes back to the steady state. Consider the rate equations

$$\frac{dn}{dt} = fc\sigma\Delta Nn - \frac{n}{T_{\mathrm{c}}}, \tag{7.1}$$

$$\frac{d\Delta N}{dt} = -2c\sigma\Delta Nn - \frac{\Delta N}{T_1} + \gamma, \tag{7.2}$$

where f is the filling factor. When the laser is at the steady state, $dn/dt = d\Delta N/dt = 0$. We have

$$N_0 = \frac{\gamma}{2c\sigma n_0 + 1/T_1} = \frac{1}{fc\sigma T_{\mathrm{c}}}, \tag{7.3}$$

where N_0, n_0 are the steady state values of ΔN and n respectively. Writing ΔN as $N_0 + N_1(t)$ and photon density n as $n_0 + n_1(t)$, and substitute back into Eqs. (7.1) and (7.2), we have

$$\frac{dn_1}{dt} = fc\sigma N_1 n_0 \tag{7.4}$$

$$\frac{dN_1}{dt} = -2c\sigma N_1 n_0 - 2c\sigma N_0 n_1 - \frac{N_1}{T_1} \tag{7.5}$$

Eqs. (7.4) and (7.5) are the linearized version of Eq. (7.1) and (7.2). Combining these two equations, we have

$$\begin{aligned}\frac{d^2 n_1}{dt^2} &= fc\sigma n_0\left[-\left(2c\sigma n_0+\frac{1}{T_1}\right)N_1-2c\sigma N_0 n_1\right]\\ &= -(\gamma fc\sigma T_\mathrm{c})\frac{dn_1}{dt}-\frac{\gamma fc\sigma T_\mathrm{c}-1/T_1}{T_\mathrm{c}}n_1.\end{aligned}\tag{7.6}$$

Define a dimensionless quantity $r \equiv \gamma fc\sigma T_\mathrm{c}T_1 = \gamma T_1/N_0$, we have

$$\frac{d^2 n_1}{dt^2} = -\frac{r}{T_1}\frac{dn_1}{dt}-\frac{r-1}{T_1 T_\mathrm{c}}n_1.\tag{7.7}$$

Note that $r-1 = 2c\sigma n_0 T_1 = 2c\sigma n_0 N_0/(N_0/T_1)$ is the ratio of the stimulated emission and the spontaneous emission rates. Looking at it differently, $r = \gamma/(1/fc\sigma T_\mathrm{c}T_1)$ is the ratio of the steady state pump rate and the threshold pump rate. In most lasers r is of the order 1-10. The solution of Eq. (7.7) is $n_1(t) = n_1(0)\exp(\eta t)$, where

$$\eta = \frac{-r/T_1 \pm \sqrt{r^2/T_1^2 - 4(r-1)/(T_1 T_\mathrm{c})}}{2}\tag{7.8}$$

In the case when $T_1/T_\mathrm{c} \gg r^2/4(r-1)$, the photon density deviation oscillates like an underdamped oscillator with a frequency

$$\omega_r = \sqrt{\frac{r-1}{T_1 T_\mathrm{c}}}.\tag{7.9}$$

It is called relaxation oscillation. The amplitude of the oscillation decays as $\exp(-rt/2T_1)$. For solid state lasers $T_1 \approx 10^{-4}$ sec, while for dye lasers, semiconductor lasers and excimer lasers $T_1 \approx 10^{-9}$ sec. Comparing with a typical $T_\mathrm{c} \approx 10^{-7}$ sec, we can see why relaxation oscillation is much more common in solid state lasers than in other lasers. For solid state lasers the period of the relaxation oscillation is on the order of microsecond. When $T_1/T_\mathrm{c} \ll r^2/(r-1)$, as it is the case for dye lasers etc., there will be no oscillations because Eq. (7.7) is overdamped.

7.2 Q-Switching

Q-switching is a technique which is widely used to generate high power optical pulses. In a laser with long upper state life time, large amount of energy can

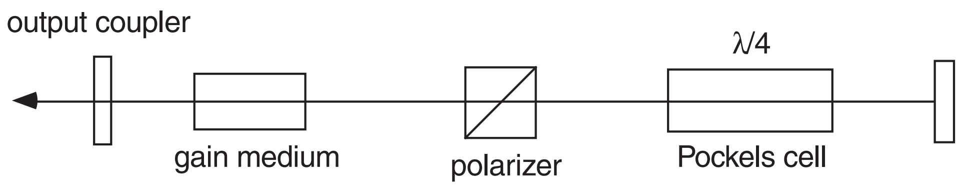

Figure 7.1: Structure of a Q-switched laser .

be stored in the gain medium if the laser is kept from lasing. From Eq. (7.2), the energy storage is $\gamma\hbar\omega T_1 f V$, where V is the mode volume of the resonator. In Q-switching the population is pumped several times above the threshold inversion while the loss of the cavity is kept very large to inhibit lasing. Then suddenly the loss is removed. Due to the large gain, a high energy pulse is quickly produced and in the mean time the population is drained by the pulse. Usually the cavity loss (cavity Q) is controlled (switched) by intracavity loss modulators, such as Pockels cells or an acousto-optic modulators. The evolution of the optical pulse and the population inversion are again governed by the nonlinear equations (7.1) and (7.2). The analysis in the last section cannot be applied because the deviations from the steady states are large. Nevertheless, it is not difficult to obtain numerical solutions by numerical integration.

Without going through numerical integrations, it is still possible to understand Q-switching qualitatively by analytic methods. Because the loss switching and the production of the pulse happen in a much shorter time scale than the pumping rate and T_1, we may omit the last two terms in Eq. (7.2). Dividing Eq. (7.1) by (7.2), we have

$$\frac{dn}{d\Delta N} = -\frac{f}{2}\left(1 - \frac{\Delta N_t}{\Delta N}\right), \tag{7.10}$$

where ΔN_t is the threshold population inversion. Equation (7.10) can be integrated with the initial condition $\Delta N = \Delta N_i$.

$$n(\Delta N) = \frac{f}{2}\left[\Delta N_t \ln\left(\frac{\Delta N}{\Delta N_i}\right) + \Delta N_i - \Delta N\right]. \tag{7.11}$$

The peak photon density occurs when $\Delta N = \Delta N_t$. The peak intensity of the pulse is

$$\begin{aligned} I_{\text{peak}} &= n_{\text{peak}}\hbar\omega c \\ &= \frac{f\hbar\omega c}{2}\left[\Delta N_t \ln\left(\frac{\Delta N_t}{\Delta N_i}\right) + \Delta N_i - \Delta N_t\right]. \end{aligned} \tag{7.12}$$

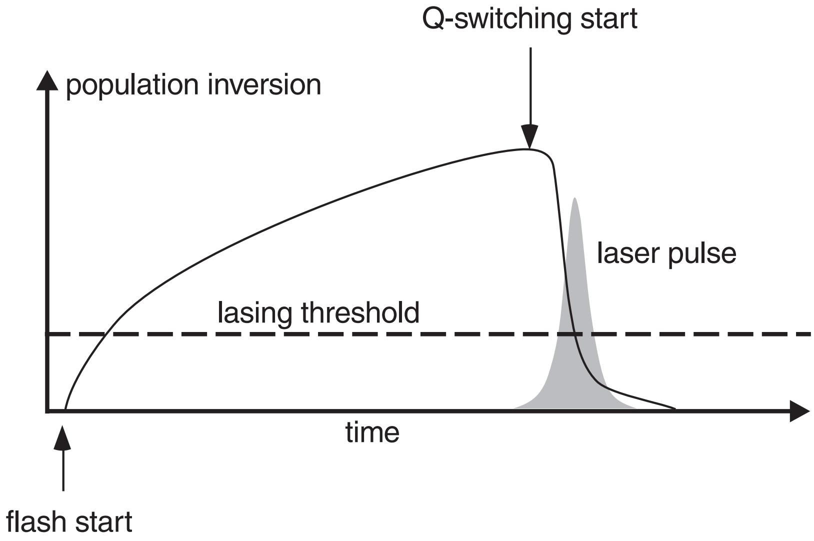

Figure 7.2: Population and photon density dynamics of a flashlamp pumped Q-switched laser .

The final population inversion can be calculated by setting Eq. (7.11) to zero.

$$\Delta N_f - \Delta N_i = \Delta N_t \ln\left(\frac{\Delta N_f}{\Delta N_i}\right). \tag{7.13}$$

The energy fluence the Q-switched pulse can be determined from the change of population inversion

$$F = \frac{l\hbar\omega}{2}(\Delta N_i - \Delta N_f), \tag{7.14}$$

where l is the length of the gain medium. The pulse duration follows from Eqs. (7.12) and (7.14).

$$t_p \approx \frac{F}{I_{\text{peak}}} = \frac{T_r}{2}\left\{\frac{\Delta N_i - \Delta N_f}{\Delta N_i - \Delta N_t\left[1 - \ln\left(\Delta N_t/\Delta N_i\right)\right]}\right\}, \tag{7.15}$$

where $T_r = 2l/(fc)$ is the round-trip time. It is seen from Eq. (7.13) that even if ΔN_i is only a few times of ΔN_t, ΔN_f becomes negligible. Under this condition Eq. (7.15) depends only on the ratio $\Delta N_i/\Delta N_t$. The pulse duration decreases from $3.3T_r$ to $0.94T_r$ as $\Delta N_i/\Delta N_t$ increases from 2 to 6.

Exercise 7.1. Design a Q-switched Nd:YAG laser which produces maximum peak output power. $T_1 = 230\ \mu$s and $\sigma = 6.5 \times 10^{-19}$ cm^2 for the Nd atoms in the YAG host crystal. The Nd:YAG crystal is 6 cm in length and 6 mm in diameter. Assume the laser is pumped by a flash lamp of 60 Joules per flash, and the pump efficiency is 8%. The cavity length of the laser is 0.5 m, and the loss in the cavity is 5%. Determine the output coupler reflectivity. Use computer graphic method if necessary. What are the output pulse energy and duration?

Although flashlamp pumped, Q-switched solid state lasers can produce gigantic laser pulses (up to Joule level) in several nanosecond, the low repetition rate of the flashlamp sometimes limits their applications. Alternatively, many solid lasers can be pumped by cw arc lamps. In such cases, high repetition rate Q-switch can be obtained by compromising both pulse energy and duration. Typically, for Nd:YAG lasers switching at 1 KHz, the pulse energy is around several millijoules and the duration is about 100 ns.

Exercise 7.2. Design a cw Q-switched Nd:YAG laser which produces maximum peak output power. $T_1 = 230\ \mu$s and $\sigma = 6.5 \times 10^{-19}$ cm^2 for the Nd atoms in the YAG host crystal. The Nd:YAG crystal is 6 cm in length and 6 mm in diameter. Assume the laser is pumped by an arc lamp of 2 kW and the pump efficiency is 8%. The Q-switch is running at 1 kHz. The cavity length of the laser is 1 m and the loss in the cavity is 5%. Determine the output coupler reflectivity. Use computer graphic method if necessary. What are the output pulse energy and duration?

Chapter 8

Pulse Propagation through Linear Media

8.1 Pulse Width and Chirp

Before we discuss principles and techniques for producing ultrashort optical pulses, let us first examine the general phenomenon of optical pulse propagation in linear media. For mathematical simplicity, let us describe the instantaneous power of an optical pulse with the Gaussian function.

$$I(t) = I_0 \exp\left(\frac{-t^2}{\tau^2}\right), \tag{8.1}$$

where τ represents the pulse width. The optical field can then be written as

$$A(t) = A_0 \exp\left(\frac{-t^2}{2\tau^2}\right) e^{-i\phi(t) - i\omega_0 t}, \tag{8.2}$$

where $\phi(t)$ represents the time dependent phase shift and ω_0 the central frequency. Assume $\phi(t)$ is a slowly varying function compared with the instantaneous power, then one may expand $\phi(t)$ in the neighborhood of $t = 0$,

$$A(t) = A_0 \exp\left(\frac{-t^2}{2\tau^2}\right) \exp\left(\frac{-i\eta t^2}{2}\right) e^{-i\omega_0 t}, \tag{8.3}$$

where we have omitted the unimportant constant term in the expansion, and the term $\phi' t$ has been merged into $\exp(-i\omega_0 t)$ to become a new central frequency $\omega_0' = \omega_0 + \phi'$. The quantity $\eta = \phi''$ is called the chirp of the pulse. It represents a linear frequency sweeping across the pulse.

Although η does not affect $I(t)$, it broadens the power spectrum.

$$\begin{aligned}\tilde{A}(\omega) &= \frac{1}{\sqrt{2\pi}} \int A(t) e^{i\omega t}\, dt \\ &= \frac{A_0}{\sqrt{1/\tau^2 + i\eta}} \exp\left[\frac{-(\omega-\omega_0)^2}{2/\tau^2 + 2i\eta}\right], \end{aligned} \tag{8.4}$$

$$|\tilde{A}(\omega)|^2 \propto \exp\left[\frac{-(\omega-\omega_0)^2}{\Delta\omega^2}\right], \tag{8.5}$$

where

$$\Delta\omega = \sqrt{\frac{1/\tau^4 + \eta^2}{1/\tau^2}}. \tag{8.6}$$

When $\eta \ll 1/\tau^2, \Delta\omega \approx 1/\tau$ as expected. But when $\eta \gg 1/\tau^2$, counter-intuitively, $\Delta\omega \approx \eta\tau$. It turns out that in such cases the major contribution to the bandwidth is the chirp η. Because phase modulation on small amplitude has little effect on the bandwidth, small pulse width helps to reduce the effect of chirp. Define the chirp-equivalent bandwidth $\Delta\omega_{\mathrm{c}} = |\eta|\tau$, one has

$$\Delta\omega = \sqrt{\frac{1}{\tau^2} + \Delta\omega_{\mathrm{c}}^2}. \tag{8.7}$$

8.2 Bandpass Filtering and Group-Delay Dispersion

As an optical pulse passes a linear medium, the medium imposes frequency-dependent optical loss and phase shift on the pulse. For mathematical simplicity, let us model the transfer function $T(\omega)$ by

$$T(\omega) = \exp\left[\frac{-(\omega-\omega_0)^2}{2\omega_g^2}\right] e^{-i\theta(\omega)}, \tag{8.8}$$

where ω_g reprsents the bandwidth of the medium, $\theta(\omega)$ the frequency-dependent phase shift, and ω_0 the central frequency of the pulse. Assume within the bandwidth of the optical pulse $\theta(\omega)$ depends weakly on ω, one may expand $\theta(\omega)$ up to the quadratic term,

$$T(\omega) = \exp\left[\frac{-(\omega-\omega_0)^2}{2\omega_g^2}\right] \exp\left[ic(\omega-\omega_0) + i\frac{D}{2}(\omega-\omega_0)^2\right]. \tag{8.9}$$

where c is called the group delay, and D the group-delay dispersion. Their physical meaning will be explained in section 8.3. After the pulse has passed the medium, the waveform becomes

$$\begin{aligned} A'(t) &= \frac{1}{\sqrt{2\pi}} \int T(\omega)\tilde{A}(\omega) e^{-i\omega t}\, d\omega \\ &= \frac{A_0}{\sqrt{1/\tau^2 + i\eta}} \frac{1}{\sqrt{h + i\kappa}} \cdot \\ &\qquad \exp\left[\frac{-(t-c)^2}{2(h+i\kappa)}\right] e^{-i\omega_0 t}, \end{aligned} \tag{8.10}$$

where

$$h = \frac{1}{(\eta^2\tau^2 + 1/\tau^2)} + \frac{1}{\omega_g^2} \tag{8.11}$$

$$\kappa = -D - \frac{\eta}{\eta^2 + 1/\tau^4}, \tag{8.12}$$

and the new chirp η' and τ' are

$$\eta' = \frac{-\kappa}{h^2 + \kappa^2} \qquad \tau'^2 = \frac{h^2 + \kappa^2}{h}, \tag{8.13}$$

respectively. Eq. (8.10) shows that the constant c equals to the delay of the pulse as it passes the medium. Terms that affect the shape of the pulse are ω_g and D.

Consider a chirpless Gaussian pulse of width τ propagating through a lossless medium ($\omega_g \to \infty$). The pulse width becomes

$$\tau' = \sqrt{\frac{\tau^4 + D^2}{\tau^2}}. \tag{8.14}$$

Define a dispersion-equivalent pulsewidth $t_d = |D|/\tau$, one has

$$\tau' = \sqrt{\tau^2 + {t_d}^2}. \tag{8.15}$$

If the pulse is chirped initially, the pulse width becomes

$$\tau' = \sqrt{\tau^2(D\eta + 1)^2 + {t_d}^2}. \tag{8.16}$$

Minimum pulse width t_d is reached when $D\eta = -1$.

Exercise 8.1. Consider an Gaussian pulse of width τ and chirp η propagating through a dispersionless medium of bandwidth ω_g. Show that under the conditions $\eta \gg 1/\tau^2$ and $\tau \gg 1/\omega_g$, it is possible that the output pulsewidth becomes shorter than the input pulsewidth.

8.3 Analogy to the Paraxial Wave Equation

In a lossless medium the evolution of the amplitude of an optical pulse is governed by the same paraxial wave equation discussed in 5.1. Consider a general waveform

$$A(z,t) = \frac{1}{\sqrt{2\pi}} \int \tilde{A}(\omega) e^{i\beta(\omega)z - i\omega t}\, d\omega, \tag{8.17}$$

where z is the coodinate in the propagation direction, and $\beta = kn$ is the effective propagation constant. The frequency dependence of β comes from the frequency dependent index of refraction $n(\omega)$. Assume that within the bandwidth of the pulse, β depends weakly on ω, one may expand β up to the quadratic term,

$$A(z,t) = \frac{1}{\sqrt{2\pi}} \int \tilde{A}(\omega) \exp\left\{ i \left[\beta_0 + \beta'(\omega - \omega_0) + \frac{\beta''}{2}(\omega - \omega_0)^2 \right] z - i\omega t \right\} d\omega. \tag{8.18}$$

The slow varying amplitude of $A(z,t)$ is $a(z,t) = A(z,t) e^{-i\beta_0 z + i\omega_0 t}$,

$$\begin{aligned} a(z,t) \;&=\; \frac{1}{\sqrt{2\pi}} \int \tilde{A}(\omega) \exp\left\{ i \left[\beta'(\omega - \omega_0) + \right.\right. \\ &\qquad \left.\left. \frac{\beta''}{2}\, (\omega - \omega_0)^2 \right] z - i(\omega - \omega_0)t \right\} d\omega. \end{aligned} \tag{8.19}$$

$$\frac{\partial a(z,t)}{\partial z} = \left[-\beta' \frac{\partial}{\partial t} - i \frac{\beta''}{2} \frac{\partial^2}{\partial t^2} \right] a(z,t). \tag{8.20}$$

Change variables

$$t' = t - \frac{z}{v_g}, \tag{8.21}$$

$$\zeta = z, \tag{8.22}$$

where $v_g = 1/\beta'$, one has

$$i \frac{\beta''}{2} \frac{\partial^2 a(\zeta, t')}{\partial t'^2} + \frac{\partial a(\zeta, t')}{\partial \zeta} = 0. \tag{8.23}$$

This is exactly the paraxial wave equation in one dimension. Eqs. (8.21)–(8.23) reveals that the pulse travels as a group with a velocity v_g, and if $\beta'' = 0$, the pulse will not change its shape.

The quantity c in section 8.2 is related to β' by $c = \beta' l$, where l is the length of the medium. Because the pulse travels with a velocity $v_g = 1/\beta'$, $c = l/v_g$ is the delay of the pulse. That is why c is called the group delay, and $D = dc/d\omega$ the group-delay dispersion.

Chapter 9

Active Mode-Locking

9.1 Concept of Mode-Locking

For many lasers the bandwidth of the laser transition is much larger than the spacing of the cavity modes. In such cases it is possible to make the laser oscillate at many cavity modes simultaneously. Let us take dye lasers as an example. The gain bandwidth of typical laser dyes is several tens of nanometers, while the cavity mode spacing is about 100 MHz. The gain bandwidth thus easily covers 10^5 cavity modes. An even more dramatic example is the Ti:sapphire laser. The bandwidth of the Ti:sapphire gain medium is about 300 nm, exceeding the bandwidth of most dielectric mirrors. If mirrors of 200-nm bandwidth are used, the laser can support over a million cavity modes. Even for the Nd:YAG laser, which is considered to be a narrowband laser for its mere 0.45-nm bandwidth, the bandwidth still covers over 1000 cavity modes.

From the relation $\Delta\nu = 1/T_r$, where $\Delta\nu$ is the cavity mode spacing and T_r the round trip time, it is clear that to produce pulses much shorter than the round trip time the laser needs to oscillate at quite a few logitudinal modes simultaneously. From the data discussed above it may seem not difficult to make broadband lasers oscillate at many modes simultaneously. However, due to mode competition, usually only a small fraction of the cavity modes covered by the gain profile survive. It takes only a little more gain for some modes to drive other modes to extinguishment. Besides, even if the gain profile can be made sufficiently flat for the laser to oscillate simutaneously in

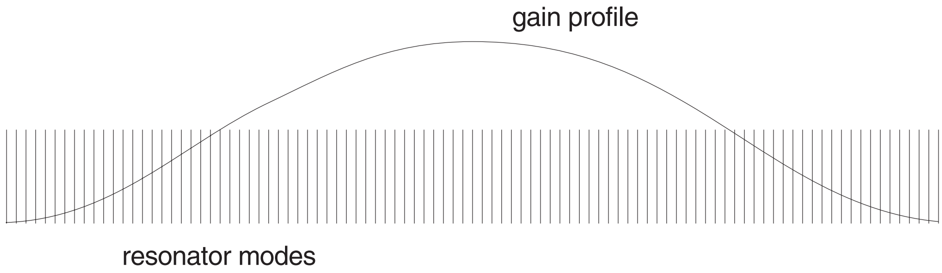

Figure 9.1: For broadband media there are many cavity modes within the gain profile.

many modes, in order to produce short pulses, the phases of the optical fields of these modes must maintain a definite relation. Otherwise the output is just a random phase superposition of fields of various frequencies. Its temporal property will not be much different from a cw incoherent white light source.

A way to make the laser oscillate at many modes with a definite phase relation between them is by coupling the modes together. The coupling locks the amplitudes and phases of the modes to a definite relation, therefore the technique is generally referred as mode-locking. A rough estimation of the minimum pulse duration achievable by mode-locking is T_r/n, where n is the number of longitudinal modes locked together. For dye lasers the limit is several tens of femtosecond, and for the Ti:sapphire laser the limit is about 10 fs, which corresponds to only four optical cycles. Even the narrowband Nd:YAG laser can produce 10-ps pulses by mode-locking.

There are two catagories of mode-locking techniques, active mode-locking and passive mode-locking. In active mode-locking intracavity amplitude or phase modulators are used to force the cavity modes to couple. Whereas in passive mode-locking the coupling is done by optical nonlinear effects in the cavity. In general, active mode-locking along is not sufficient to produce ultrashort pulses, because as the pulse shortens it sees less and less modulation. In contrast, nonlinear effects increase with the instantaneous power, therefore the effects become stronger and stronger as the pulse shortens. For this reason passive mode-locking is more effective in producing ultrashort pulses. But in cases where the pulse duration is limited by the bandwidth to the extent that nonlinear effects are weak, active mode-locking is a more reliable choice. For the same reason, during the starting stage of passive mode-locking, when the pulse is not short enough to take advantage of the nonlinear effects, active mode-locking can be a starting method for passive mode-locking. In this chapter we shall discuss the basics of active mode-locking. Passive mode-locking will be discussed in chapter 11.

9.2 Amplitude Modulation and Phase Modulation

Before we discuss active mode-locking, let us describe briefly two types of intracavity modulators. The first one is the electro-optic modulator, which is usually used as a phase modulator. The second one is the acousto-optic modulator, which is usually used as an amplitude modulator.

A phase modulator is usually made of a crystal with large electro-optic coefficient. In such crystals the index of refraction changes linearly with the strength of an externally applied electric field E,

$$n(E) = n_0 + \alpha E. \tag{9.1}$$

For a sinusoidal electric field $E = E_0 \cos(\Omega t)$, the depth of the phase modulation is $k\alpha E_0 l$, where l is the lenght of the modulator.

A typical acousto-optic modulator is made of a small block of optical glass or crystal. At the ends of the block, piezoelectric transducers are attached. The transducers are driven by sinusoidal electrical signal to produce acoustic waves in the solid. Counter-propagating waves add up to form a standing wave. Usually the driving electrical signal is tuned to an acoustic resonance frequency of the block to enhance the intensity of the standing wave. Because acoustic wave deforms the solid, the deformation causes a modulation of index of refraction proportional to the amplitude of the acoustic wave.

$$\Delta n(x,t) = \beta \cos(k_a x) \cos(\omega_a t). \tag{9.2}$$

where k_a, ω_a are the propagation number and angular frequency of the acoustic wave, respectively, and β^2 is proportional to the intensity of the acoustic wave. The index modulation acts as an oscillating phase grating to deflect the incoming light. If the deflection angle is sufficiently large, the deflected light will escape from the laser cavity. The index modulation thereby produces an amplitude modulation on the transmitted light. In order to produce a sufficiently large deflection angle, the wavelength of the acoustic wave must not be too large compared with the optical wavelength. This can be achieved by driving the modulator at radio-frequencies. For example, the speed of sound in fused silica is about 6000 m/s. At 40 MHz the wavelength is 150 μm. For light of 1-μm wavelength the deflection angle is approximately $1/150 \approx 0.4^\circ$.

The modulation depth can be calculated by analyzing the coupling between the incoming light and the deflected light through the phase grating. Because the length of the modulator is only a few centimeters, the time for the light to pass the modulator is much shorter than the period of the modulation frequency. Under this condition we only need to consider a static phase grating in the analysis. Assume the amplitudes of both the incoming light and the deflected light change slowly due to coupling as they propagate in the z-direction through the modulator.

$$E_i(\mathbf{r}) = A_i(z)\exp(i\mathbf{k}_i \cdot \mathbf{r}), \tag{9.3}$$

$$E_{d\pm}(\mathbf{r}) = A_{d\pm}(z)\exp(i\mathbf{k}_{d\pm} \cdot \mathbf{r}), \tag{9.4}$$

where $\mathbf{k}_i = k\hat{z}$ and $\mathbf{k}_{d\pm} = k\hat{z} \pm k_a\hat{x}$. For simplicity let us assume that the electric fields are in the $\hat{y}$ direction. Substituting $E = E_i + E_{d+} + E_{d-}$ into the Maxwell equation

$$\nabla \times \nabla \times E + \frac{n^2\omega^2}{c^2}E = 0, \tag{9.5}$$

omitting small terms such as Δn^2, $\partial^2 A_i/\partial z^2$, $\partial^2 A_{d\pm}/\partial z^2$, k_a^2, and $k_a \partial A_{d\pm}/\partial z$, and collecting terms of $\exp(ikz)$ and $\exp[i(kz \pm k_a x)]$ we have

$$\frac{\partial A_{d\pm}}{\partial z} = -i\kappa A_z \tag{9.6}$$

$$\frac{\partial A_z}{\partial z} = -i(\kappa A_{d+} + \kappa A_{d-}), \tag{9.7}$$

where $\kappa = nk\beta\cos(\omega_a t)/2$. The solution is

$$A_i(z) = A_i(0)\cos(\kappa z), \tag{9.8}$$

$$A_{d\pm}(z) = -iA_i(0)\sin(\kappa z). \tag{9.9}$$

When κl is small, where l is the length of the modulator,

$$A_i(l) = A_i(0)\left(1 - \frac{\kappa^2 l^2}{2}\right). \tag{9.10}$$

It is seen that the amplitude is modulated periodically at $2\omega_a$, the depth of modulation is $n^2k^2l^2\beta^2/16$.

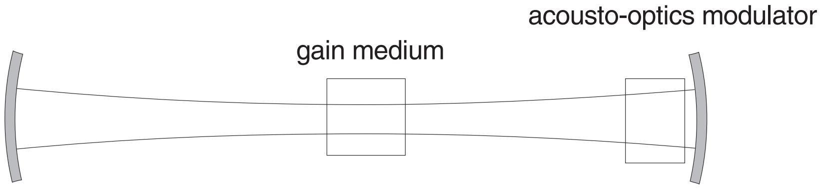

Figure 9.2: Basic structure of active mode-locking lasers.

9.3 Self-Consistent Solutions

Consider a broadband laser with an intracavity sinusoidal loss modulator. when the frequency of the modulation is synchronized with the cavity round trip time, the laser repetitively amplifies the portion of light that passes the modulator when the loss is minimum. Therefore the light forms pulses centered at the $t = t_0 + nT_r$, where t_0 is the time of minimum loss. Because the wings of the pulses always experience more loss than the peak, the pulses becomes shorter and shorter. But this pulse shortening process cannot continue indefinitely, because as the bandwidth of the pulses becomes comparable to that of the gain medium, the gain medium becomes a spectral filter. The spectral center of the pulses experiences more gain than the spectral wings. Therefore the pulses are broadened temporally by the gain medium. When the two counter-acting effects balance each other in the steady state , the laser produces a steady train of identical pulses.

We shall now analyze the temporal waveform of the output pulses. Let us assume the bandwidth of the steady state pulses is much smaller than the bandwidth of the gain medium, and the pulse width is much smaller than the period of loss modulation. Under these conditions only the shape of the gain profile near the peak and the shape of the loss modulation near the minuimum are important. In other words, we may expand the gain profile $G(\omega) = \exp[g(\omega)L]$, where L is the length of the gain medium, around the peak and the loss waveform $l(t)$ around the minimum:

$$G(\omega) = G_0\left[1 - \frac{(\omega-\omega_0)^2}{2\omega_g^2}\right] \approx G_0 \exp\left[\frac{-(\omega-\omega_0)^2}{2\omega_g^2}\right], \tag{9.11}$$

$$l(t) = l_0\left[1 - \frac{(t-t_0)^2}{2T_a^2}\right] \approx l_0 \exp\left[\frac{-(t-t_0)^2}{2T_a^2}\right]. \tag{9.12}$$

From the fact that both the Fourier transform and the product of Gaussian functions are Gaussian, it is not difficult to see that by assuming the steady state waveform is a Gaussian function, a self-consistent solution can be obtained. Let $E(t) = E_0 \exp(-t^2/2\tau^2)\exp(-i\omega_0 t)$, the self-consistent condition reads

$$E(t) = \text{I.F.T.}\{\text{F.T.}\{E(t)l(t)\} \times G(\omega)\}, \tag{9.13}$$

where F.T. and I.F.T. denote the Fourier transform and its inverse respectively. Using the relation

$$\text{F.T.}\left\{\exp(-at^2)\right\} = \sqrt{\frac{1}{2a}}\exp\left(\frac{-\omega^2}{4a}\right) \tag{9.14}$$

and choosing $t_0 = 0$,we obtain

$$\tau^2 = \frac{\tau^2 T_a^2}{\tau^2 + T_a^2} + \frac{1}{\omega_g^2}, \tag{9.15}$$

$$G_0 l_0 = \sqrt{1 + \frac{\tau^2}{T_a^2}}. \tag{9.16}$$

Eq. (9.15) determines the pulse duration, and Eq. (9.16) determines the gain coefficient G_0. For solid sate lasers the time scale for gain recovery is on the order of the upper state lifetime T_1, which is much larger than the round trip time T_r. Therefore G_0 is related to the time-averaged intensity $\bar{I}$ by the gain saturation equation.

$$\frac{\ln G_0}{L} = \frac{g_0}{1 + \bar{I}/I_s}, \tag{9.17}$$

where g_0 is the peak small signal gain and I_s the saturation intensity. $\bar{I}$ is related to the amplitude of the pulse by

$$\bar{I} = \frac{1}{T_r}\frac{c|E_0|^2}{8\pi}\int_{-\infty}^{\infty}\exp\left(\frac{-t^2}{\tau^2}\right)\,dt = \frac{c|E_0|^2\tau}{8\sqrt{\pi}T_r}. \tag{9.18}$$

The three equations Eqs. (9.15)–(9.17) give the values of the three unknowns τ, G_0, and E_0.

For an acousto-optic modulator, $1/T_a^2 = \gamma\omega_m^2$, where $\gamma = n^2k^2l^2\beta^2/16$ is the modulation depth and $\omega_m = 2\omega_a$ the modulation frequency. Since $T_a \gg \tau$, Eq. (9.15) can be simplified to

$$\tau = \sqrt{\frac{T_a}{\omega_g}}. \tag{9.19}$$

It is seen that the pulse duration is inversely proportional to the square root of the product of ω_g, ω_m, and the r.f. power fed into the modulator.

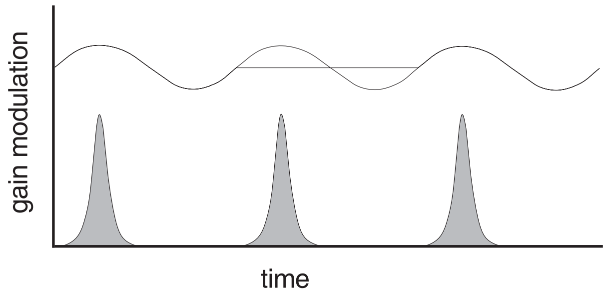

Figure 9.3: In active mode-locking, gain modulation causes pulses to form at the position of highest gain.

Exercise 9.1. Consider a Nd:YAG laser ($\lambda = 1.06\mu$m) mode-locked by amplitude modulation. The driving frequency of the acousto-optic modulator ω_a is 80 MHz, and the modulation depth is 20%. The full width at half maximum (FWHM) intensity of the fluorescent spectrum of the gain medium is 0.45 nm. Calculate the full width at half maximum intensity of the pulses.

Solution: Let us model the intensity spectrum of the fluorescence $I(\omega)$ as

$$I(\omega) = I_0 \exp\left[\frac{-\ln 2(\omega-\omega_0)^2}{(\Delta\omega_{\text{FWHM}}/2)^2}\right],$$

where $\Delta\omega_{\text{FWHM}} = 2\pi c/\lambda^2 \times 0.45\times 10^{-9} \approx 7.5\times 10^{11}/\text{sec}$. $\omega_g = \Delta\omega_{\text{FWHM}}/(2\sqrt{\ln 2})$.

$$1/\tau = \sqrt{\sqrt{0.2}\cdot 2\pi \cdot 2\cdot 8\times 10^7 \cdot \omega_g}.$$

$$\tau_{\text{FWHM}} = \tau \cdot 2\sqrt{\ln 2} \approx 100 \text{ ps}.$$

Exercise 9.2. (1) For a laser mode-locked by phase modulation, derive a general expression for the pulse duration τ similar to Eq. (9.19). (2) Consider a Nd:YAG laser ($\lambda = 1.06\mu$m) mode-locked by phase modulation. The modulation frequency Ω is 80 MHz, and the modulation depth is 0.2 radian. The full width at half maximum (FWHM) intensity of the fluorescent spectrum of the gain medium is 0.45 nm. Calculate the full width at half maximum intensity of the pulses.

9.4 General Solutions

A general equation for the waveform $E(t) = a(t)\exp(-i\omega_0 t)$ that satisfies Eq. (9.13) can be derived by using the Taylor expansions Eqs. (9.11) and (9.12).

$$a(t) = \frac{1}{\sqrt{2\pi}} \int G_0 \left[1 - \frac{(\omega-\omega_0)^2}{2\omega_g^2}\right] \tilde{b}(\omega) \exp\left[-i(\omega-\omega_0)t\right] d\omega, \tag{9.20}$$

where

$$\tilde{b}(\omega) = \frac{1}{\sqrt{2\pi}} \int l_0 \left(1 - \frac{t^2}{2T_a^2}\right) a(t) \exp\left[i(\omega-\omega_0)t\right] dt. \tag{9.21}$$

Eqs. (9.20) and (9.21) can be written as

$$a(t) = G_0 l_0 \left(1 + \frac{1}{2\omega_g^2}\frac{\partial^2}{\partial t^2}\right)\left(1 - \frac{t^2}{2T_a^2}\right) a(t). \tag{9.22}$$

Keeping only the first-order terms, we have

$$\left(\frac{1}{2\omega_g^2}\frac{\partial^2}{\partial t^2} - \frac{t^2}{2T_a^2} + 1 - \frac{1}{G_0 l_0}\right) a(t) = 0. \tag{9.23}$$

Eq. (9.23) is nothing but the Schrödinger equation for the simple harmonic oscillator. The solution are Hermite-Gaussian functions.

$$a(t) = \mathrm{H}_n\left(\frac{t}{\tau}\right)\exp\left(\frac{-t^2}{2\tau^2}\right), \tag{9.24}$$

$$\tau = \sqrt{\frac{T_a}{\omega_g}}, \tag{9.25}$$

$$1 - \frac{1}{G_0 l_0} = \frac{2n+1}{2\omega_g T_a}. \tag{9.26}$$

Because Hermite-Gaussians form a complete set, any solution of Eq. (9.23) is a superposition of them. From Eq. (9.26) it is seen G_0 is larger for larger n, which means higher order solutions are less favored than the "ground state" solution ($n = 0$) in gain competition. Therefore in practice only the $n = 0$ solution is important. Note that because $\omega_g T_a \gg 1$, Eq. (9.26) is equivalent to Eq. (9.16).

Exercise 9.3. Generalize the theory above by including intracavity group-delay dispersion and phase modulation.

Chapter 10

Pulse Propagation through Nonlinear Media

10.1 Pulse Propagation through Saturable Media

An optical pulse is amplified as it propagates through a gain medium. At the same time it drains the population inversion, thereby reduces the gain. Because gain depletion occurs during the interaction of the pulse with the gain medium, the result is that the front portion of the pulse experiences more gain than the rear portion, hence the front portion is amplified more. The effect, known as dynamic gain saturation, is nonlinear because it depends on the intensity of the pulse. Similarly, as an optical pulse propagates through an absorptive medium, it depletes the ground state of the medium, thereby reduces the absorption. The front portion of the pulse is absorbed more than the rear portion. In this section we discuss dynamic saturation of gain and absorption caused by an optical pulse, and conversely, the distortion of pulse shape caused by dynamic saturation.

Let us start from the Maxwell equation, Eq. (4.31). For solutions of the form $E(z,t) = E'(z)\exp(-i\omega t)$, Eq. (4.31) can be written as

$$\frac{\partial^2 E'}{\partial z^2} + \frac{\omega^2 E'}{c^2} = -\frac{4\pi\omega^2}{c^2}\chi(\omega)E', \tag{10.1}$$

Using $n^2 = 1 + 4\pi\mathrm{Re}\chi$, where n is the index of refraction, and $c\sigma\Delta N = -4\pi\omega\mathrm{Im}\chi$, one has

$$\frac{\partial^2 E'}{\partial z^2} + \frac{n^2\omega^2 E'}{c^2} = \frac{i\omega\sigma\Delta N}{c}E', \tag{10.2}$$

The solution of Eq. (10.2) is $E'(z) = E_0 \exp[i\beta(\omega)z + \alpha(\omega)z]$, where $\beta - i\alpha = \sqrt{n^2\omega^2/c^2 - i\omega\sigma\Delta N/c} \approx n\omega/c - i\sigma\Delta N/(2n)$. Thus the general solution of the Maxwell equation is $E(z,t) = (1/\sqrt{2\pi}) \int \tilde{A}(\omega) \exp[i\beta(\omega)z + \alpha(\omega)z - i\omega t]\, d\omega$. The slow varying amplitude of $E(z,t)$ is $a(z,t) = E(z,t) \exp[-i\beta(\omega_0)z + i\omega_0 t]$. Assuming group velocity dispersion is negligible, then

$$a(z,t) = \frac{1}{\sqrt{2\pi}} \int \tilde{A}(\omega) \exp\left[i\frac{d\beta}{d\omega}(\omega - \omega_0)z + \frac{\sigma\Delta N}{2n}z - i(\omega - \omega_0)t\right] d\omega \tag{10.3}$$

satisfy

$$\frac{\partial a(z,t)}{\partial z} + \frac{1}{v_g}\frac{\partial a(z,t)}{\partial t} = \frac{\sigma\Delta N}{2n} a(z,t), \tag{10.4}$$

where $1/v_g = d\beta/d\omega$ is the group velocity. Multiply Eq. (10.4) by $a^*(z,t)$ and add its complex conjugate, one has

$$\frac{\partial I(z,t)}{\partial z} + \frac{1}{v_g}\frac{\partial I(z,t)}{\partial t} = \frac{\sigma}{n}\Delta N(z,t) I(z,t), \tag{10.5}$$

where $I(z,t)$ is the intensity. Now we shall examine the other equation that govern the change of population inversion. Assuming the time the pulse propagate through the gain medium is so short that the population relaxation terms in Eq. (4.41) is negligible, one has

$$\frac{\partial \Delta N(z,t)}{\partial t} = -\frac{a\sigma}{\hbar\omega}\Delta N(z,t) I(z,t), \tag{10.6}$$

where $a = 2$ if the lower state lifetime is much longer than pulse duration, and $a = 1$ for the opposite case. Eqs. (10.5) and (10.6) are the pair of rate equations that govern the pulse amplification and population depletion in the gain medium. By a change of variable $t' = t - z/v_g$, Eq. (10.5) can be simplified to

$$\frac{\partial I(z,t')}{\partial z} = \frac{\sigma}{n}\Delta N(z,t') I(z,t'), \tag{10.7}$$

Note that t' represents the relative position of the pulse in time. For example, if the length of a rectangular pulse in space is l, the difference between the front edge and rear edge in t' is always l/v_g. Integrating Eq. (10.7) over the length of the gain medium L, one has

$$I(L,t') = G(t') I(0,t'), \tag{10.8}$$

where $G(t') = \exp[\sigma\overline{\Delta N}(t')L/n]$ and

$$\overline{\Delta N}(t') = \frac{1}{L}\int_0^L \Delta N(z,t')dz. \tag{10.9}$$

$G(t')$ and $\overline{\Delta N}(t')$ are the saturated gain of the amplifier and the average polulation inversion respectively, seen by the portion of the pulse at t'. Integrating Eq. (10.6) over the length of the gain medium L and substituting Eq. (10.7) in, one has

$$\begin{aligned} \frac{d\overline{\Delta N}(t')}{dt'} &= -\frac{an}{\hbar\omega L}\left[I(L,t') - I(0,t')\right] \\ &= -\frac{an}{\hbar\omega L}\left\{\exp\left[\sigma\overline{\Delta N}(t')L/n\right] - 1\right\} I(0,t') \\ &= -\frac{an}{\hbar\omega L}\left\{1 - \exp\left[-\sigma\overline{\Delta N}(t')L/n\right]\right\} I(L,t'). \end{aligned} \tag{10.10}$$

Let us define accumulated energy fluences $U(z,t') \equiv \int_{t'_0}^{t'} I(z,r)dr$, where t'_0 is sufficiently earlier than the pulse center that nothing has entered the gain medium yet, and integrate Eq. (10.10). One has

$$U(0,t') = \frac{\hbar\omega}{a\sigma}\ln\left[\frac{1 - 1/G_0}{1 - 1/G(t')}\right], \tag{10.11}$$

$$U(L,t') = \frac{\hbar\omega}{a\sigma}\ln\left[\frac{G_0 - 1}{G(t') - 1}\right], \tag{10.12}$$

where $G_0 \equiv \exp[\sigma\overline{\Delta N}(t_0)L/n]$ is the unsaturated gain. The time-dependent gain that changes the shape of the pulse is related to the accumulated energy fluences by

$$G(t') = \frac{G_0}{G_0 - (G_0 - 1)\exp\left[-U(0,t')/U_s\right]}, \tag{10.13}$$

or

$$G(t') = 1 + (G_0 - 1)\exp\left[-U(L,t')/U_s\right], \tag{10.14}$$

where $U_s \equiv \hbar\omega/(a\sigma)$. At a time t' sufficiently late that the pulse has gone through the gain medium completely, one has $U(0,t') = U_{\text{in}}$ and $U(L,t') = U_{\text{out}}$, where U_{in} and U_{out} are the input fluence and output fluence respectively. Define energy gain $G_E \equiv U_{\text{out}}/U_{\text{in}}$, then from Eqs. (10.11) and (10.12),

$$G_E = \frac{\ln\left\{(G_0 - 1)/[G(t') - 1]\right\}}{\ln\left\{(1 - 1/G_0)/[1 - 1/G(t')]\right\}}. \tag{10.15}$$

Substitute Eq. (10.13) for $G(t')$, one obtains

$$G_E = \frac{U_s}{U_{\text{in}}}\ln\left\{1 + \left[\exp\left(\frac{U_{\text{in}}}{U_s}\right) - 1\right]G_0\right\}. \tag{10.16}$$

In the above analysis we have concentrated on the dynamics of pulse amplification or absorption. Now let us turn the attention to the population dynamics. The equation that governs the population change is

$$\frac{\partial \Delta N(z,t)}{\partial t} = -ac\sigma \Delta N(z,t) n(z,t) - \frac{\Delta N(z,t) - \Delta N_0(z)}{T_1}, \tag{10.17}$$

where $\Delta N_0(z)$ is the steady state population density before the pulse entering the medium. To solve Eq. (10.17), let us assume $\Delta N_0(z) = 0$ first, and define $\Delta N'(z,t) \equiv \Delta N(z,t) \exp(t/T_1)$. Eq. (10.17) becomes

$$\frac{\partial \Delta N'(z,t)}{\partial t} = -ac\sigma \Delta N'(z,t) n(z,t). \tag{10.18}$$

The solution is

$$\Delta N'(z,t) = \Delta N'(z,t_0) \exp\left(-ac\sigma \int_{t_0}^{t} n(z,s)\, ds\right), \tag{10.19}$$

where t_0 is an early time when the pulse has not entered the medium yet. For the case of $\Delta N_0(z) \neq 0$, the solution can be obtained from Eq. (10.21) by the method of variation of parameters. Let us replace $\Delta N'(z,t_0)$ by $N_v(z,t)$ and substitute it into Eq. (10.17). One has

$$\frac{\partial \Delta N_v(z,t)}{\partial t} \exp\left(-ac\sigma \int_{t_0}^{t} n(z,s)\, ds\right) = \frac{\Delta N_0(z)}{T_1}, \tag{10.20}$$

Eq. (10.18) is known as the Debye relaxation equation. One has

$$\begin{aligned} \frac{\partial}{\partial t'}\left[\exp\left(\frac{t'}{T_1}\right)\Delta N(z,t')\right] &= \exp\left(\frac{t'}{T_1}\right)\left[\frac{\partial \Delta N(z,t')}{\partial t'} + \frac{\Delta N(z,t')}{T_1}\right] \\ &= \exp\left(\frac{t'}{T_1}\right)\left[-ac\sigma \Delta N(z,t_0) n(z,t')\right]. \end{aligned} \tag{10.21}$$

Integrating Eq. (10.21) over t' one obtains the formal solution of $\Delta N(z,t')$,

$$\Delta N(z,t') = \Delta N(z,t_0)\left[1 - \frac{a\sigma}{\hbar\omega}\int_{-\infty}^{t'} I(z,s) \exp\left(-\frac{t'-s}{T_1}\right) ds\right], \tag{10.22}$$

where $I(z,t') = cn(z,t')\hbar\omega$ is the instantaneous intensity of the pulse. Note that if the pulse duration is much shorter than T_1, immediately after the pulse propagates through the gain medium at t_1, the population inversion is reduced to

$$\Delta N(z,t_1) = \Delta N(z,t_0)\left[1 - \frac{a\sigma}{\hbar\omega} U(z,t_1)\right], \tag{10.23}$$

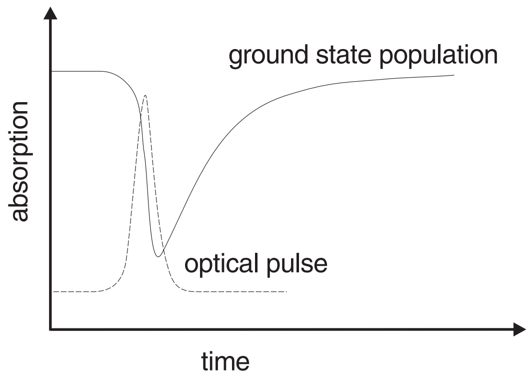

Figure 10.1: Response of saturable absorbers to optical pulses.

The time scale of the gain depletion equals approximately to the pulse duration. After that the population inversion relaxes back to its steady state value according to

$$\Delta N(z,t') = \Delta N(z,t_0) + [\Delta N(z,t_1) - \Delta N(z,t_0)] \exp\left(-\frac{t'-t_1}{T_1}\right). \quad (10.24)$$

The time scale of the relaxation is T_1. On the contrary, if the pulse duration is much larger then T_1, we may treat the factor $\exp[(s-t')/T_1]$ as the δ-function $T_1\delta(t'-s)$. Then Eq. (10.22) reduces to

$$\Delta N(z,t') = \Delta N(z,t_0)\left[1 - \frac{a\sigma T_1}{\hbar\omega} I(z,t')\right], \quad (10.25)$$

which is consistent with Eq. (1.58) for the cw case.

10.2 Self-Phase Modulation and Optical Soliton

The index of refraction comes from the dipole response of a material to the electric field. Nonlinearity of the dipole response causes the index of refraction to change slightly with the intensity of light. The effect is called the optical Kerr effect.

$$n(I) = n_0 + n_2 I, \quad (10.26)$$

where n_2 is called the nonlinear index of refraction. In most transparent materials n_2 is very small, that is why the effect becomes significant only when the material interacts with high power laser pulses.

As an optical pulse propagates through a Kerr medium, the Kerr effect produces a rapid varying phase modulation across the pulse envelope. It causes the pulse center to have a larger phase shift than the pulse wings. The phenomenon is called self-phase modulation. The equation governing the propagation of an optical pulse in a dispersive Kerr medium can be derived by introducing the phase modulation $\kappa|a(z,t)|^2$ into Eq. (8.20). Eq. (8.23) becomes

$$-\frac{\beta''}{2}\frac{\partial^2 a(z,t)}{\partial t^2} + \kappa|a(z,t)|^2 a(z,t) + i\frac{\partial a(z,t)}{\partial z} = 0. \tag{10.27}$$

If $a(z,t)$ represents the envelope of the electric field, then $\kappa = kn_2c/8\pi$. Eq. (10.27) is called the nonlinear Schrödinger equation, where $\kappa|a(z,t)|^2$ serves as the nonlinear potential energy. The equation belongs to a special class of nonlinear partial differential equations that can be solved by the so called inverse scattering method. Under the condition that β'' and κ have the opposite sign, Eq. (10.27) has the following simple solution of a pulse:

$$a(z,t) = \sqrt{\left|\frac{\beta''}{\kappa}\right|}\frac{1}{\tau}\mathrm{sech}\left(\frac{t}{\tau}\right)\exp\left(\frac{-i\beta'' z}{2\tau^2}\right), \tag{10.28}$$

where τ represents the pulse width. Note that the amplitude is inversely proportional to the pulse width. If the initial pulse shape is made according to Eq. (10.28), then the pulse will not change its shape during the propagation, in contrast to propagation in a linear dispersive medium. For the particular waveform described by Eq. (10.28), the effect of dispersion is balanced off by the self-pahse modulation. This is the simplest example of an optical soliton.

10.3 Pulse Compression

Self-phase modulation can be used to compress an optical pulse. Let us consider a Gaussian pulse propagating through a dispersionless Kerr medium. The electric field of the pulse can be written as

$$E(t) = E_0 \exp\left(\frac{-t^2}{2\tau^2}\right)e^{i\phi(t)-i\omega_0 t}, \tag{10.29}$$

where

$$\phi(t) = kn_2 l I_0 \exp\left(\frac{-t^2}{\tau^2}\right), \tag{10.30}$$

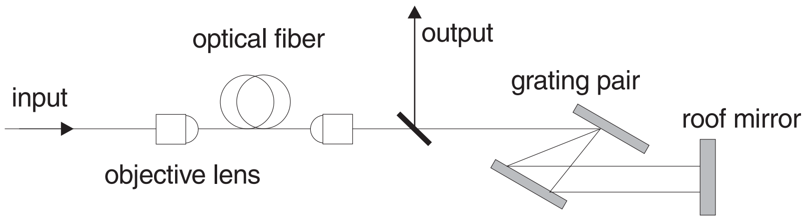

Figure 10.2: Basic structure of a pulse compressor.

l is the length of the medium, and $I_0 = cE_0^2/8\pi$ the peak intensity of the pulse. For pedagogical reasons let us "approximate" the Gaussian waveform $\exp(-t^2/\tau^2)$ by a parabola $1 - ht^2/\tau^2$, where h is a constant. If we choose $h = 1$, $1 - ht^2/\tau^2$ crosses zero at $t = \tau$, then the phase modulation is over estimated. On the contrary if we let $h = 1/4$, $1-ht^2/\tau^2$ crosses zero at $t = 2\tau$, then the phase modulation is under estimated. Therefore $0.25 < h < 1$. The approximation is far from being accurate, however, as we shall see such approximation enable us to gain much insight into the phenomenon of pulse compression because it greatly simplifies the mathematical analysis. After the pulse is self-phase modulated, we send it through a dispersive medium. According to Eq. (8.16), if we choose the thickness l of the dispersive medium to be

$$L\beta'' = -\frac{\tau^2}{2hkn_2lI_0}, \tag{10.31}$$

then the pulse duration becomes

$$\tau' = \left|\frac{\tau}{2hkn_2lI_0}\right|. \tag{10.32}$$

If one makes l sufficiently large, τ' can be made as small as desired. Of course, such wonderful thing happens only becaues we used an unrealistic parabolic approximation to the Gaussian phase modulation. In reality the pulse cannot be compressed without limit. The limit is determined by the higher order phase modulation (terms proportional to t^3 and up) and higher order dispersion (terms proportional to $(\omega - \omega_0)^3$ and up).

10.4 Self-Focusing

Similar to self-phase modulation, optical Kerr effect causes the center of a Gaussian beam to have a larger phase than the edge. It makes the phase velocity of the beam center slower than that of the edge, which resembles focusing by a lens. The phenomenon is called self-focusing. Consider the wave equation in a dielectric medium

$$\nabla^2 A + \frac{\omega^2 n^2}{c^2} A = 0, \tag{10.33}$$

where A is the vector potential which is transverse to the z-axis. Under the condition $n_2 I \ll n_0$, the wave equation is approximately

$$\nabla^2 A + \frac{\omega^2 n_0^2}{c^2} A = -\frac{2\omega^2 n_0 n_2 I}{c^2} A. \tag{10.34}$$

Under the paraxial approximation

$$A(x, y, z) = u(x, y, z) e^{ik_0 z}, \tag{10.35}$$

where $k_0 = \omega n_0 / c$, and that $I \approx p|A|^2$, where p is a proportion constant, Eq. (10.35) becomes

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 2ik_0 \frac{\partial u}{\partial z} + 2k_0^2 \frac{n_2}{n_0} p|u|^2 u = 0. \tag{10.36}$$

Other than the trivial plane wave, no simple analytical solution is known for Eq. (10.36). However, if the wave is confined in a two-dimemsional space, the corresponding equation

$$\frac{\partial^2 u}{\partial x^2} + 2ik_0 \frac{\partial u}{\partial z} + 2k_0^2 \frac{n_2}{n_0} p|u|^2 u = 0 \tag{10.37}$$

is the nonlinear Shrödinger equation. For $n_2 > 0$ it has a simple solution in which the beam size remains the same during the propagation.

$$u(x, z) = \sqrt{\frac{n_0}{n_2 p}} \frac{1}{k_0 w} \mathrm{sech}\left(\frac{x}{w}\right) \exp\left(\frac{iz}{2k_0 w^2}\right), \tag{10.38}$$

where w represents the beam size. Note that the amplitude is inversely proportional to the beam size. Analogous to the optical soliton in time domain, the effect of diffraction is balanced off by self-focusing. The solution in Eq. (10.38) is called spatial soliton.

If we assume self-focusing is sufficiently weak such that for a Gaussian input beam the spatial profile remains close to Gaussian. Then we may again use the pedagogical approximation discussed in section 10.3 for the Gaussian phase modulation in Eq. (10.36). Namely, $I_0 \exp(-2r^2/w^2) \approx$

$I_0(1 - 2hr^2/w^2)$, where r is the lateral distance from the beam center and h is a proportional constant. As mentioned before, the approximation is far from being accurate, but it enables us to gain much insight into the physics by greatly simplifying the mathematics. Now consider the Kerr medium as a series of n lenses separated by a distance l/n, where l is the thickness of the medium. The focal length of each lens is a function of the intensity of the beam at the position of the lens. Using this model, we can adapt the $ABCD$ matrices and the q-parameter to describe the self-focusing effect.

Consider the intensity profile of a Gaussian beam

$$I(r) = \frac{2P}{\pi w^2} \exp\left(-\frac{2r^2}{w^2}\right), \tag{10.39}$$

where P is the power of the beam. The phase shift due to self-focusing is equal to $-(kn\Delta z -$ constant term), *i.e.*,

$$\phi(r, z) = -kn_2[I - I(0)]\Delta z \approx \frac{4hkn_2Pr^2}{\pi w(z)^4}\Delta z, \tag{10.40}$$

where $\Delta z = l/n$. Taking into account the diffraction of the beam after propagating a distance Δz and ignoring the Δz^2 term, we may write the $ABCD$ matrix as

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & \frac{\Delta z}{n_0} \\ -\frac{8hn_2P}{\pi w(z)^4}\Delta z & 1 \end{bmatrix}. \tag{10.41}$$

Let $Q(z) = 1/q(z)$, where $q(z)$ is the q-parameter of the Gaussian beam, we have

$$\begin{aligned} Q(z + \Delta z) - Q(z) &= \frac{Q(z) + C}{BQ(z) + 1} - Q(z) \approx C - B[Q(z)]^2 \\ &= -\left\{\frac{8\pi hPn_2}{\lambda^2}[\mathrm{Im}Q(z)]^2 + \frac{[Q(z)]^2}{n_0}\right\}\Delta z, \end{aligned} \tag{10.42}$$

where we have used the relation $\mathrm{Im}Q = -\lambda/(\pi w^2)$. Let $n \to \infty$, Eq. (10.42) becomes a differential equation governing the evolution of $Q(z)$. By a simple change of variable $\mathrm{Re}Q + i\mathrm{Im}Q \to \mathrm{Re}Q + i\alpha\mathrm{Im}Q \equiv \tilde{Q}$ where

$$\alpha = \sqrt{1 - \frac{8\pi hPn_2n_0}{\lambda^2}}, \tag{10.43}$$

Eq. (10.42) becomes

$$\frac{d\tilde{Q}(z)}{dz} = -\frac{\left[\tilde{Q}(z)\right]^2}{n_0}. \tag{10.44}$$

The solution is

$$\frac{1}{\tilde{Q}(z)} - \frac{1}{\tilde{Q}(0)} = \frac{z}{n_0}. \tag{10.45}$$

Note that this is the same solution for $Q(z)$ when $n_2 = 0$. Therefore, the behavior of a weakly self-focused Gaussian beam looks the same as without self-focusing if we use the "normalized" beam size $w/\sqrt{\alpha}$ to replace w.

Chapter 11

Passive Mode-Locking

11.1 Mode-Locking by Fast Saturable Absorbers

In a laser cavity a fast saturable absorber produces an intensity dependent round-trip loss factor $l(t)$ of the following form

$$l(t) = l_0 \left[1 + \alpha |a(t)|^2\right], \tag{11.1}$$

where $a(t)$ is the temporal envelope of the electric field and α a proportional constant. The saturable absorber makes the loss of the pulse center smaller than the pulse wings, thereby shortens the pulse in successive round trips. Note that in Eq. (11.1) the round-trip loss factor is defined in a silimar way as the round-trip gain. Larger loss factor represents smaller loss of optical power. Let us assume that the only pulse broadening effect counter-acting the pulse shortening effect is the bandwidth limit of the gain medium. Then the equation for the waveform evolution is

$$\Delta a(t) = \frac{e^{i\phi}}{\sqrt{2\pi}} \int G(\omega)\tilde{b}(\omega) \exp\left[-i(\omega-\omega_0)t\right] d\omega - a(t), \tag{11.2}$$

$$\tilde{b}(\omega) = \frac{1}{\sqrt{2\pi}} \int l(t)a(t) \exp\left[i(\omega-\omega_0)t\right] dt, \tag{11.3}$$

where $a(t)$ is the temporal envelope of the pulse, $G(\omega)$ is the frequency dependent round-trip gain, ω_0 is the center frequency of the pulse, ϕ is the phase shift per round trip, and $\Delta a(t)$ represents the change of $a(t)$ per round trip. If L is the length of the gain medium, $G(\omega) = \exp(gL)$. Assume the bandwidth of the pulse is much smaller than that of the gain medium, we may expand $g(\omega)$ around ω_0.

$$\Delta a(t) = \frac{e^{i\phi}}{\sqrt{2\pi}} \int G_0 \left[1 - \frac{(\omega-\omega_0)^2}{2\omega_g^2}\right] \tilde{b}(\omega) \exp\left[-i(\omega-\omega_0)t\right] d\omega - a(t). \quad (11.4)$$

Note that in the expansion of $G(\omega)$ we assume the center frequency of the pulse coincides with the peak of the gain. Eqs. (11.3) and (11.4) can be written as

$$\Delta a(t) = e^{i\phi} G_0 l_0 \left(1 + \frac{1}{2\omega_g^2}\frac{\partial^2}{\partial t^2}\right)\left[1 + \alpha |a(t)|^2\right] a(t) - a(t). \quad (11.5)$$

In the steady state, $\Delta a(t) = 0$. Keeping only the first-order changes, we have

$$\left[\frac{1}{2\omega_g^2}\frac{\partial^2}{\partial t^2} + \alpha|a(t)|^2 + 1 - \frac{e^{-i\phi}}{G_0 l_0}\right] a(t) = 0. \quad (11.6)$$

There exists a simple solution

$$a(t) = a_0 \mathrm{sech}\left(\frac{t}{\tau}\right), \quad (11.7)$$

where

$$\tau = \sqrt{\frac{1}{\alpha}}\frac{1}{\omega_g |a_0|}, \quad (11.8)$$

$$|a_0| = \sqrt{\frac{2}{\alpha}\left(\frac{1}{G_0 l_0} - 1\right)}, \quad (11.9)$$

and

$$\phi = 0. \quad (11.10)$$

Note that the pulse duration τ is inversely proportional to the gain bandwidth ω_g, unlike in active mode-locking where $\tau \propto \sqrt{1/\omega_g}$.

At this point it is not difficult to see that if we include the effects of self-phase modulation and dispersion, Eq. (11.6) becomes

$$\left[(B - iD/2)\frac{\partial^2}{\partial t^2} + (\alpha + i\gamma)|a(t)|^2 + 1 - \frac{e^{-i\phi}}{G_0 l_0}\right] a(t) = 0, \quad (11.11)$$

where $B = 1/(2\omega_g^2)$ represents the bandwidth limiting effect, D and γ are the group-delay dispersion and the self-phase modulation per round trip. Eq. (11.11) also has a simple solution

$$a(t) = a_0 \left[\mathrm{sech}\left(\frac{t}{\tau}\right)\right]^{1+i\beta}, \quad (11.12)$$

where

$$\frac{B - iD/2}{\tau^2}(2 + 3i\beta - \beta^2) = (\alpha + i\gamma)|a_0|^2, \tag{11.13}$$

$$\frac{B - iD/2}{\tau^2}(1 + i\beta)^2 + 1 - \frac{e^{-i\phi}}{G_0 l_0} = 0. \tag{11.14}$$

Eliminating τ^2, we have

$$\left(\frac{e^{-i\phi}}{G_0 l_0} - 1\right)\frac{2 + i\beta}{1 + i\beta} = (\alpha + i\gamma)|a_0|^2. \tag{11.15}$$

For solid sate lasers the time scale for gain recovery is on the order of the upper state lifetime T_1, which is much larger than the round trip time T_r. Therefore G_0 is related to the time-averaged intensity $\bar{I}$ by the gain saturation equation.

$$\frac{\ln(G_0)}{L} = \frac{g_0}{1 + \bar{I}/I_s}, \tag{11.16}$$

where g_0 is the peak small signal gain and I_s the saturation intensity. $\bar{I}$ is related to the amplitude of the pulse by

$$\bar{I} = \frac{1}{T_r}\frac{c|a_0|^2}{8\pi}\int_{-\infty}^{\infty} \operatorname{sech}^2\left(\frac{t}{\tau}\right)\, dt = \frac{c|a_0|^2\tau}{4\pi T_r}. \tag{11.17}$$

The two complex equations Eqs. (11.13) and (11.14) plus the real equation Eq. (11.16) give the values of the five unknowns τ, β, ϕ, G_0 and $|a_0|$.

11.2 Additive-Pulse Mode-Locking

In section 11.1 it is seen that intracavity fast saturable absorbers amplify energy fluctuation in a cw laser to produce short pulses. Although real absorbers such as dye solutions can be used for this purpose, it is not the best idea. Not only because dye molecules do not last long under intense radiation, but also because the shortest upper state life time of dye molecules is limited to the order of 10 ps. This means the model breaks down for pulse duration smaller than 10 ps. In order to produce femtosecond laser pulses, one must use nonlinear effects which are nonresonant, so that the response time is not limited by the upper state lifetime.

It is shown in sec 10.2 that optical Kerr effect produces a phase shift proportional to intensity. If this phase modulation can be translated to amplitude modulation, we have a equivalent fast saturable absorber. One

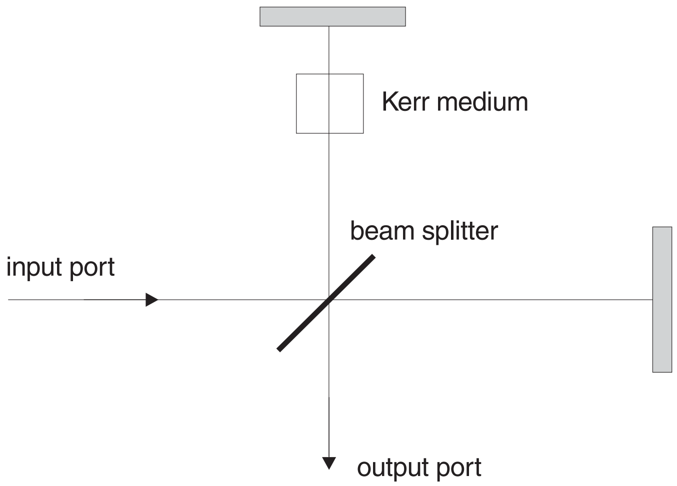

Figure 11.1: Nonlinear Michelson Interferometer for pulse shortening.

way of doing it is by an interferometer. Consider a Michelson interferometer in which one of the arms contains a Kerr medium, as shown in Fig. 11.1. At the nonreturning output port, the intensity is

$$\begin{aligned} I_o &= \left| RTE_i + RTE_i e^{i(\phi_d+\phi_k)} \right|^2 \\ &= \left| RTE_i \right|^2 \left[2 + 2\cos(\phi_d + \phi_k) \right], \end{aligned} \tag{11.18}$$

where E_i is the input field, R, T the complex reflection and transmission coefficients for the field respectively ($|R|^2 + |T|^2 = 1$), ϕ_d the phase delay due to optical path difference, and ϕ_k the self-phase modulation. Because ψ_k is a small number proportional to the input intensity, the Taylor expansion of Eq. (11.18) is

$$\frac{I_o}{I_i} = 2|RT|^2 \left[1 + \cos(\phi_d) - \sin(\phi_d) k n_2 l I_i \right], \tag{11.19}$$

where l is the medium length. This is of the same form as Eq. (11.1). The fast saturable-absorber effect is strongest when $\phi_d = -\pi/2$.

The basic structure of additive-pulse mode-locking lasers consists of two coupled cavities. The main cavity contains the gain medium, the auxiliary cavity contains the Kerr medium and the output coupler. Interferience occurs at a beam splitter that couples the two cavity. The laser can be viewed as an intracavity interferometer with a nonlinear phase modulator. Usually a short piece of optical fiber is utilized as the phase modulator. Self-phase

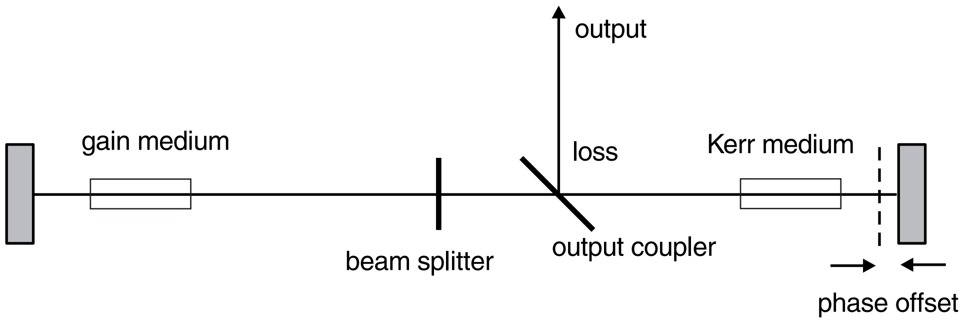

Figure 11.2: Basic structure of additive-pulse mode-locking lasers.

modulation in the fiber produces a phase difference between the peak and the wings of the pulse. This phase modulation is converted into an amplitude modulation at the beam splitter. When the lengths of the two cavities are properly set, this amplitude modulation favors the growth of the peak, thereby modelocks the laser.

In the following analysis, we assume the coupled-cavity laser is already pulsed. Initial pulses may come from mode beating or intracavity intensity fluctuation. Since the initial pulses or intensity fluctuation are much longer than the optical cycle, one can write the electric field as $E(t-z/c)\exp[i\omega(t-z/c)]$, where $E(t-z/c)$ is a slow varying function, z and t are spatial and time coordinate respectively, or simply $E(\tau)\exp(i\omega\tau)$, where τ is the reduced time, measuring the distance from the pulse center. For simplicity in notation, let us replace τ with t, with the understanding that now t is a coordinate to describe the relative position of a moving waveform. To solve the equation of motion, we separate the electric field into two parts, $A(t)\exp(iwt)$, the field in the main laser cavity, and $B(t)\exp(it)$, the field in the external coupled cavity, propagating toward the beam splitter. $A(t)$ and $B(t)$ are both slow varying and carry the phase of the fields. The two fields $A(t)$ and $B(t)$ add at the beam-splitter once each round trip. Since in the starting stage the pulse is evolving, $A(t)$ and $B(t)$ are different on each round trip. Therefore we may label them $A_n(t)$ and $B_n(t)$, according to the number of round trips, n. The equation of motion of two empty coupled cavities with matched cavity lengths is

$$\begin{bmatrix} A_{n+1}(t) \\ B_{n+1}(t) \end{bmatrix} = \begin{bmatrix} R & iT \\ iT & R \end{bmatrix} \begin{bmatrix} A_n(t) \\ B_n(t) \end{bmatrix}, \tag{11.20}$$

where R is the square root of power reflectivity of the beam splitter, and T the square root of power transmission. Note that the phase difference between the reflected wave and the transmitted wave is always $\pi/2$, otherwise the law of energy will be broken at the beam splitter. If now the gain is added to the main cavity, and the loss to the auxiliary cavity, and a slight cavity length mismatch is introduced, then the equation of motion becomes (see Fig. 11.2)

$$\begin{bmatrix} A_{n+1}(t) \\ B_{n+1}(t) \end{bmatrix} = \begin{bmatrix} RG & iTG \\ iTle^{i\phi} & Rle^{i\phi} \end{bmatrix} \begin{bmatrix} A_n(t) \\ B_n(t) \end{bmatrix}, \tag{11.21}$$

where G is the square root of power gain, $1 - l^2$ the power loss and ϕ the phase difference due to the slight cavity length mismatch. Eq. (11.21) can be solved with the standard eigenvalue method. Let λ_1 and λ_2 be the two eigenvalues of the transformation matrix in Eq. (11.21), its solution can be written as

$$\begin{bmatrix} C_{n+1}(t) \\ D_{n+1}(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \begin{bmatrix} C_n(t) \\ D_n(t) \end{bmatrix} \tag{11.22}$$

with

$$\lambda = \frac{R(G + le^{i\phi}) \pm \sqrt{R^2(G + le^{i\phi})^2 - 4Gle^{i\phi}}}{2}, \tag{11.23}$$

or equivalently,

$$C_n(t) = \lambda_1^n C_0(t), \qquad D_n(t) = \lambda_2^n D_0(t), \tag{11.24}$$

where $C_n(t), D_n(t)$ are linear combinations of $A_n(t), B_n(t)$, and can be thought of as the two eigenmodes of the coupled cavities. It can be shown easily on a computer that, for practical values of R, l, and G, the absolute value of one of the λ_s is larger than 1, while the other is smaller than 1. Therefore one of the eigenmodes, say $D_n(t)$, is damped, while the other will grow with time. Although $|\lambda_1|$, the gain of $C_n(t)$, is larger than 1 initially, it will drop to 1 in the steady state because of gain saturation in the laser medium. In other words, as the energy of the pulse grows higher and higher, g will drop according to Eq. (1.58).

$$G = \exp\left(\frac{g_0 L}{1 + I_a/I_s} - \alpha L\right) \tag{11.25}$$

till $|\lambda_1| = 1$. In Eq. (11.25), g_0 is the small signal gain coefficient, I_a the average intensity in the gain medium and I_s the saturation intensity. The term αL represents the absorption of the gain medium at lasing wavelength, with α the absorption coefficient and L the thickness of the gain medium.

When G is adjusted to make $|\lambda_1| = 1$, the numerical fact that $|\lambda_2|$ is smaller than one does not change. In Eq. (11.25) it is assumed that the gain cross section is small, so that the gain saturates slowly comparing with pulse duration and round trip time. Therefore, G has no dependence on t. This is a good approximation for most of the solid state lasers because of their small emission cross sections.

Now we shall take into consideration the self-phase modulation which is a function of t. The phase difference ϕ becomes $\phi_0 + \kappa I(t)$, where ϕ_0 is the phase difference due to slight cavity length difference, while $\kappa I(t)$ is due to the self-phase modulation and proportional to the instantaneous intensity $I(t)$. The parameter κ is positive and proportional to the length of the Kerr medium and the nonlinear index of refraction n_2. Since the round trip gain λ_1 depends on t through ϕ, different portions of the pulse have different amplitude gain. This differential gain is the origin of pulse formation and shortening. At an operating point where $|\lambda_1|$ increases with ϕ, the gain of the pulse peak will be larger than the pulse wings. In this case the energy in the pulse will gradually shift to the peak, and the pulse shortens.

In Fig. 11.3 the mode-locking parameter $\partial|\lambda_1|\partial\phi$ and g are plotted as a function of ϕ_0 for a typical additive-pulse mode-locking laser. Between zero and π the $\partial|\lambda_1|\partial\phi$ is positive, whereas it is negative between $-\pi$ and zero. In this region the laser favors the cw mode instead of pulsing modes. In the region where $\partial|\lambda_1|\partial\phi > 0$, the slope of g is negative. Because g is a monotonically decreasing function of pulse energy, one can use a feedback circuit which monitors the average power to lock ϕ_0 at the region where $\partial|\lambda_1|\partial\phi > 0$. This will prevent the laser from drifting away from the mode-locking region.

11.3 Kerr-Lens Mode-Locking

Additive-pulse mode-locking utilizes self-phase modulation in the temporal domain to produce a differential gain that favors high power. In Kerr-lens mode-locking, self-phase modulation in the spatial domain is utilized. As shown in section 5.5, in a low loss, stable laser resonator, the q-parameter of the fundamental Gaussian mode is determined by the following self-consistent equation $q = Aq + B/(Cq + D)$, where A, B, C, D are the matrix elements of the combined ABCD matrix over a round trip. Modes with q different from the self-consistent one suffer from larger loss, therefore will be damped out

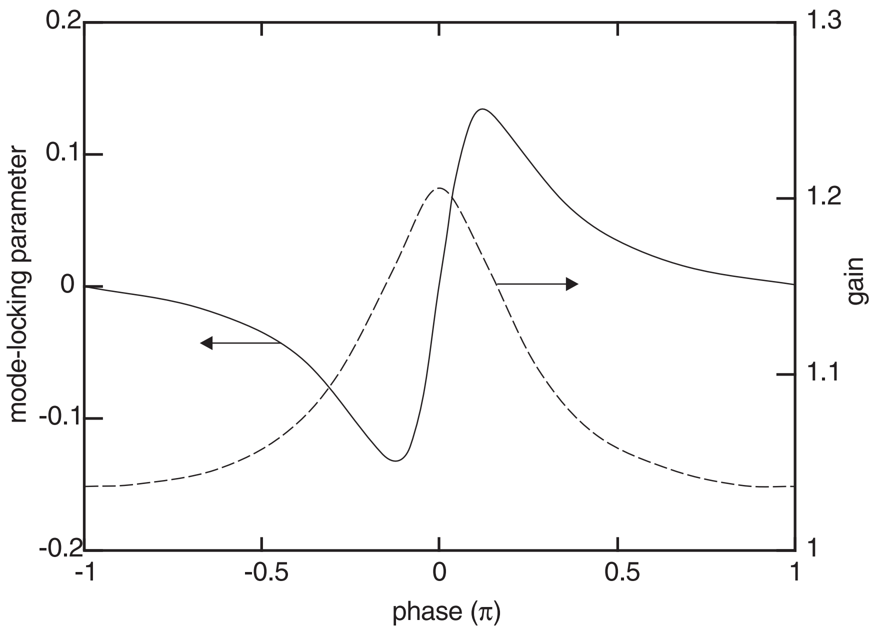

Figure 11.3: Mode-locking parameter as a function of phase for additive-pulse mode-locking.

in the long run. In a pulsed laser with intracavity self-focusing effect, the q-parameter is a function of the instantaneous power $I(t)$. In this case, it is possible to design the laser in such a way that it favors high power. Since the average power is limited by the rate equations, the only way for the laser to operate at high power is becoming mode-locked.

Figure 11.4 shows the basic structure of a Kerr-lens mode-locking laser. Self-focusing in the gain medium causes the beam size to shrink in the gain medium. Because the output coupler is at the image point of the beam waist in the gain medium, beam size at the output coupler is also smaller for higher power beam. By making the pump beam slightly smaller than usual, it will match better to a higher power beam, thereby providing a differential gain that favors high power. In the meantime one can put an aperture in front of the output coupler. The aperture causes more loss to the lower power beam, providing a differential loss that again favors higher power beam. Kerr-lens mode-locking is much simpler in construction than additive-pulse mode-locking, in particular, it does not require interferometric accuracy in alignment, therefore it has become the most popular method for generating femtosecond laser pulses.

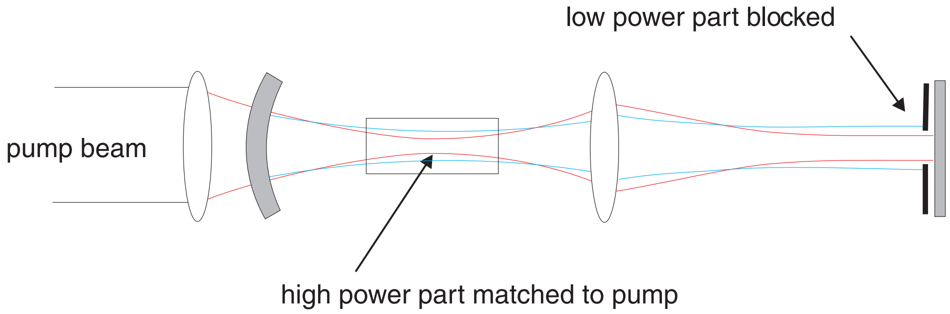

Figure 11.4: Basic structure of Kerr-lens mode-locking lasers.

Although it is not easy to calculate $q(I)$ accurately by analytical methods, under the aberationless approximation, the magnitude of the effect can be estimated by the q-parameter rescaling theory discussed in section 10.4. For simplicity, let us consider only the differential gain caused by mode matching with the pump beam. Assume both the pump beam and the laser beam are at the Gaussian M_{00} mode, and assume the temporal shape of the pulse is also Gaussian, the spatially convoluted gain G is

$$\ln G(P) = \int_0^\infty \int_{-L/2}^{L/2} \sigma \Delta N(r,z) \frac{2}{\pi w(P,z)^2} \exp\left[\frac{-2r^2}{w(P,z)^2}\right] 2\pi r \, dr dz, \quad (11.26)$$

where P the intracavity power of the laser beam, σ the emission cross section, L the length of the gain medium, and $\Delta N(r,z)$ is the population inversion density. Due to self-focusing, in Eq. (11.26) the beam size w is a function of P. The population inversion density $\Delta N(r,z)$ is proportional to the pump intensity.

$$\Delta N(r,z) = \Delta N_0(z) \frac{2}{\pi w_p(z)^2} \exp\left[\frac{-2r^2}{w_p(z)^2}\right], \quad (11.27)$$

where w_p is the size of the pump beam. In order to make the laser favor high power mode, w_p is chosen to maximize the difference between $G(P)$ and $G(0)$. After the integration over r,

$$\ln G(P) = \frac{2\sigma}{\pi} \int_{-L/2}^{L/2} \Delta N_0(z) \left[\frac{1}{w_p(z)^2 + w(P,z)^2}\right] dz. \quad (11.28)$$

When the gain medium is pumped from both end, $N_0(z)$ has a weak dependence on z. From Eq. (5.20), and the approximation that $N_0(z)$ is constant, the integration in Eq. (11.28) can be carried out.

$$\ln G(P) = \frac{4\sigma}{\pi} \frac{\Delta N_0}{U(P)} \arctan \frac{L}{2V(P)}, \quad (11.29)$$

where

$$b_p = \frac{\pi w_{p0}^2}{\lambda_p}, \qquad b(P) = \frac{\pi w_0(P)^2}{\lambda}, \tag{11.30}$$

$$U(P) = \left[w_{p0}^2 + w_0(P)^2\right]^{1/2} \left[\frac{w_{p0}^2}{b_p^2} + \frac{w_0(P)^2}{b(P)^2}\right]^{1/2}. \tag{11.31}$$

$$V(P) = \left[w_{p0}^2 + w_0(P)^2\right]^{1/2} \left[\frac{w_{p0}^2}{b_p^2} + \frac{w_0(P)^2}{b(P)^2}\right]^{-1/2}. \tag{11.32}$$

Usually matching with the pump mode is done at the center of the gain medium, which is also the place where self-focusing is most significant. One can expand the scaling relation Eq. (10.43) to the first order of P,

$$w_0(P)^2 = w_0^2\left(1 - \frac{4\pi h P n_2 n_0}{\lambda^2}\right), \tag{11.33}$$

and substitute into Eq. (11.29) to obtain the differential gain dG/dP. By adjusting the pump mode size, dG/dP can be optimized for mode-locking.

11.4 Pulse Shortening Rate

To the first order expansion, the intensity gain (or loss) of a passive mode-locking laser has the following form,

$$\Gamma(t) = \Gamma_0 + 2\alpha I(t), \tag{11.34}$$

In this section we shall investigate the relation between α and the pulse shortening rate. For simplicity, let us assume the pulse shape remains close to a Gaussian function through out the pulse forming process.

$$I_n(t) = \frac{E}{\sqrt{\pi}\tau_n} \exp\left(\frac{-t^2}{\tau_n^2}\right), \tag{11.35}$$

where $I_n(t)$ is the intensity profile, τ_n the pulse width, E the pulse energy, and n represents the number of round trips since starting. The equation of motion for the intensity profile is

$$I_{n+1}(t) = I_n(t)\left[\Gamma_n + 2\alpha I_n(t)\right], \tag{11.36}$$

where Γ_0 in Eq. (11.34) has been replaced by Γ_n to account for the change of Γ_0 with respect to n, as a result of the changing pulse shape. Assume the pulse shortening time scale is much longer than the upper state lifetime, then the laser will have reached the steady state power before the pulse shortens. In this case the pulse energy does not change from round trip to round trip,

$$E = \int I_{n+1}(t)dt = \int \left[\Gamma_n I_n(t) + 2\alpha I_n^2(t)\right] dt = E\left(\Gamma_n + \sqrt{\frac{2}{\pi}}\frac{\alpha E}{\tau_n}\right). \tag{11.37}$$

The pulse width can be obtained from the definition

$$\frac{\tau_n^2}{2} = \frac{1}{E}\int t^2 I_n(t)dt. \tag{11.38}$$

Multiply both sides of Eq. (11.36) by t^2/E, integrate, and use Eq. (11.37) to eliminate Γ_n, one obtains

$$\tau_{n+1}^2 = \tau_n^2 - \frac{\alpha E}{\sqrt{2\pi}}\tau_n. \tag{11.39}$$

Since τ_n changes only slightly after each round trip, Eq. (11.39) can be replaced by a differential equation of $\tau(n) \equiv \tau_n$.

$$2\tau(n)\frac{d\tau(n)}{dn} = -\frac{\alpha E}{\sqrt{2\pi}}\tau(n). \tag{11.40}$$

The solution is

$$\tau_n = \tau_0 - \frac{\alpha E}{2\sqrt{2\pi}}n. \tag{11.41}$$

This is the pulse width evolution as a function of the number of round trips. Since the steady state pulse width is negligible compared with the initial pulse width, the number of round trips for the pulse width to reach steady state is approximately $2\sqrt{2\pi}\tau_0/(\alpha E)$. This result point out that it is essential to have rapid fluctuation or seed pulses in the cavity, otherwise τ_0 is very large and it takes long time for the pulse to form.

Eq. (11.40) shows that the fractional pulse shortening rate $d\tau/(\tau dn)$ is inversely proportional to τ, which means the pulse shortening process becomes more and more efficient as the pulse width is reduced. This escalating trend can only be stopped by the finite bandwidth of the gain medium or the intracavity group-delay dispersion. From Eq. (8.13) it is seen that the fractional change of pulse width per round trip for a cavity with bandwidth ω_g is

$$\frac{d\tau}{dn} = \frac{1}{2\tau\omega_g^2}. \tag{11.42}$$

If bandwidth is the only major pulse broadening effect, the balance point is

$$\tau = \frac{\sqrt{2\pi}}{\alpha E \omega_g^2}. \tag{11.43}$$

This is very close to the exact result derived from Eq. (11.8),

$$\tau = \frac{2}{\alpha E \omega_g^2}. \tag{11.44}$$

The same technique can be used on active mode-locking. The equation of motion is

$$I_{n+1}(t) = I_n(t)\left(\Gamma_n - \frac{t^2}{T_a^2}\right). \tag{11.45}$$

Integrate both sides over t, one has

$$E = \int I_{n+1}(t)dt = E\left(\Gamma_n - \frac{\tau_n^2}{2T_a^2}\right). \tag{11.46}$$

Multiply both sides of Eq. (11.45) by t^2/E and integrate over t, one has

$$\tau_{n+1}^2 = \Gamma_n \tau_n^2 - \frac{\tau_n^4}{T_a^2}. \tag{11.47}$$

Eliminating Γ_n,

$$2\tau(n)\frac{d\tau(n)}{dn} = -\frac{\tau(n)^4}{2T_a^2}, \tag{11.48}$$

the solution is

$$\frac{1}{\tau_n^2} - \frac{1}{\tau_0^2} = \frac{n}{2T_a^2}. \tag{11.49}$$

Steady state pulse width is determined by

$$\frac{d\tau}{dn} = 0 = \frac{\tau^3}{4T_a^2} - \frac{1}{2\tau\omega_g^2}, \tag{11.50}$$

and the solution $\tau = 2^{1/4}\sqrt{T_a/\omega_g}$ is again very close to the exact result Eq. (9.25).

Chapter 12

Dispersion Compensation

12.1 Grating Pairs

Most optical materials such as glass and various dielectric crystals are transparent in the visible and near infrared spectral region. However, in this most useful region, the group velocity dispersion is usually positive, in other words, longer wavelength light propagates faster than shorter wavelength light. For ultrashort pulse lasers, maintaining low dispersion is extremely important, because dispersion is a major pulse broadening effect. In other applications such as pulse compression, it is essential to have a device that produces negative dispersion to compensate for the positive dispersion that comes from optical materials as well as the positive chirp that comes from self-phase modulation.

Negative group delay dispersion can be produced by forcing light of different wavelength to travel different paths. There are two ways, one is using grating pairs, the other is using prism pairs. Fig. 12.1 shows the light paths for different wavelengths. Because longer wavelength light diffracts at a larger angle, from the figure it is clear that longer wavelength light travels a shorter path. It is not difficult to calculate the path length for each wavelength using geometric optics. The group delay dispersion are

$$a = b \tag{12.1}$$

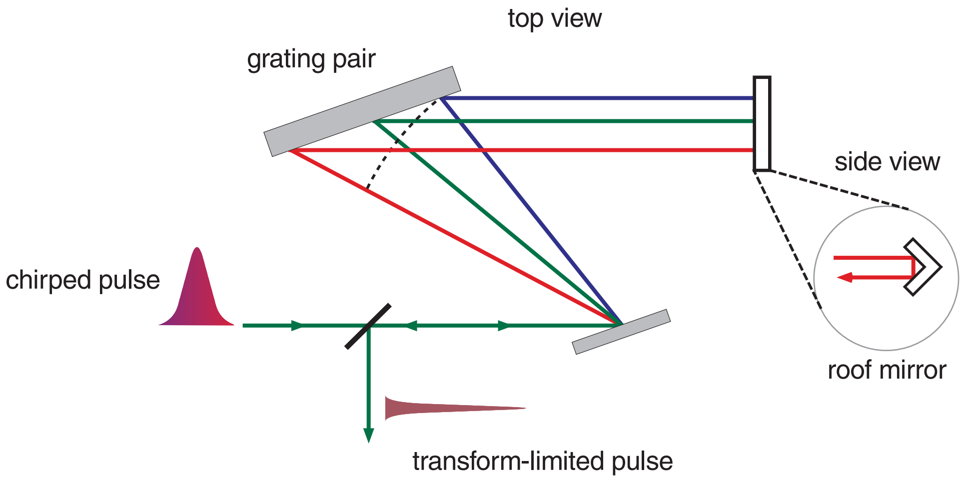

Figure 12.1: A grating pair for negative group delay dispersion.

12.2 Prism Pairs

Grating pairs can provide large negative dispersion, however, because it is not possible to eliminate the specular reflection and higher order diffraction, they are too lossy to put in the oscillator cavity. This problem can be overcome by prism pairs shown in Fig. 12.2. In a typical configuration, the prisms are cut at the Brewster angle on both sides, so that the p-wave suffers no loss. In a prism pair, longer wavelength light refracts at a small angle. At the first glance it may seem that the path length for longer wavelength is shorter. However, one should not forget that longer wavelength light goes through more glass, which as a larger index of refraction than air. Therefore more path length is gained in the prism. Again, the group delay dispersion can be calculated from geometric optics.

$$a = b \tag{12.2}$$

12.3 Chirped Mirrors

As one can see in the above sections,

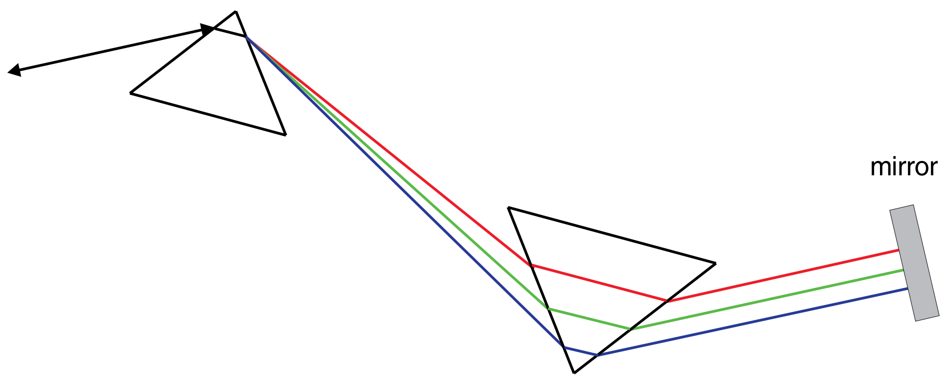

Figure 12.2: A prism pair for negative group delay dispersion.

Appendix A

Theory of Measurement Uncertainty

A.1 Variance of Arithmetic Means

In any measurement, there is always uncertainty in the result. It is only a matter of degree. There are many causes of the uncertainty, such as vibration of the floor, power fluctuation of the electricity, temperature and pressure change of the surrounding, or even the quantization error of digital recording instruments. These causes are mostly uncorrelated, therefore the measurement result can be treated as random variables. The simplest way to describe a random variable x is by its expectation value $E(x)$ and its variance $V(x)$.

$$E(x) \equiv \alpha \equiv \int xP(x)\,dx, \tag{A.1}$$

$$V(x) \equiv \sigma_x^2 \equiv E(x-\alpha)^2 = \int (x-\alpha)^2 P(x)\,dx, \tag{A.2}$$

where $P(x)$ is the probability distribution of x. The uncertainty of measurement can be reduced by taking the average of many measurements, because the deviation from the true result which can only be obtained with perfect instruments in an ideal environment tends to cancel partially for each other. It is possible to estimate how much the uncertainty is reduced. Let $x_1, x_2, \ldots, x_n$ be the results of n measurements and y the average result.

$$y = \frac{1}{n}\sum_i x_i. \tag{A.3}$$

Since the sum of random variables is also a random variable, one can ask what the expectation value and the variance of y are.

$$E(y) = \frac{1}{n} \sum_i E(x_i) = \alpha, \tag{A.4}$$

$$\begin{aligned} V(y) &= E(y-\alpha)^2 \\ &= \frac{1}{n^2} E\left(\sum_i x_i - n\alpha\right)^2 \\ &= \frac{1}{n^2} E\left[\sum_i (x_i-\alpha)^2 + \sum_{i\neq j}(x_i-\alpha)(x_j-\alpha)\right] \\ &= \frac{\sigma_x^2}{n}. \end{aligned} \tag{A.5}$$

In the above equation, because x_i and x_j are independent random variables,

$$E\left[\sum_{i\neq j}(x_i-\alpha)(x_j-\alpha)\right] = \sum_{i\neq j} E(x_i-\alpha)E(x_j-\alpha) = 0. \tag{A.6}$$

It is seen that the average measurement result has a much smaller variance when n is large. This is how averaging works.

A.2 Sample Variance

A subtle question is: how can one know σ_x^2 in the first place? To determine, one needs to know $P(x)$ precisely, which is possible only with an infinite number of measurements. However, one can still estimate σ_x^2 from the finite (but large) number of measurement results available. Let us define the sample variance s_x^2 by

$$s_x^2 \equiv \frac{1}{n} \sum_i (x_i - y)^2. \tag{A.7}$$

Note that the definition of s_x^2 does not involve $P(x)$. In fact, s_x^2 itself is a random variable whose expectation value we would like to know.

$$\begin{aligned} E(ns_x^2) &= \sum_i E(x_i-y)^2 \\ &= \sum_i E\left[(x_i-\alpha)-(y-\alpha)\right]^2 \end{aligned}$$

$$\begin{aligned}
&= \sum_i E\left[(x_i-\alpha)^2 - 2(x_i-\alpha)(y-\alpha) + (y-\alpha)^2\right] \\
&= \sum_i E(x_i-\alpha)^2 - \frac{2}{n}\sum_i E[(x_i-\alpha)(x_i-\alpha)] \\
&\quad -\frac{2}{n}\sum_{i\neq j} E[(x_i-\alpha)(x_j-\alpha)] + \sum_i E(y-\alpha)^2. \qquad \text{(A.8)}
\end{aligned}$$

The first two terms combines to be a good approximation of $(n-2)\sigma_x^2$, the third term is zero because x_i and x_j are independent, and the last term is σ_x^2 according to Eq. (A.5). Therefore one has

$$E(s_x^2) = \frac{n-1}{n}\sigma_x^2, \qquad \text{(A.9)}$$

and

$$V(y) \approx \frac{1}{n-1}s_x^2, \qquad \text{(A.10)}$$

Eq. (A.10) tells us how to plot the error bar in our presentation of measurement data.

A.3 Central Limit Theorem

In Sec. A.1, we showed how to estimate the uncertainty of the average measurement result. One can push much further to show that the probability distribution of y is a Gaussian function regardless of the shape of $P(x)$. This is known as the central limit theorem.

For simplicity let us change variable x to $x-\alpha$ such that the expectation value of x is zero. Let us also define the characteristic function $C(k)$ to be the Fourier transformation of $P(x)$,

$$C(k) = \frac{1}{\sqrt{2\pi}}\int P(x)e^{ikx}\,dx. \qquad \text{(A.11)}$$

By Taylor expansion

$$C(k) = b_0 + b_2k^2 + b_3k^3 + \cdots, \qquad \text{(A.12)}$$

where

$$b_0 = \frac{1}{\sqrt{2\pi}}\int P(x)\,dx = \frac{1}{\sqrt{2\pi}}$$

$$\begin{aligned} b_1 &= \frac{1}{\sqrt{2\pi}} \int P(x)(ix)\,dx = 0 \\ b_2 &= \frac{1}{\sqrt{2\pi}} \int P(x)\frac{(ix)^2}{2}\,dx = -\frac{\sigma_x^2}{2\sqrt{2\pi}}, \end{aligned} \tag{A.13}$$

and in general

$$b_n = \frac{1}{\sqrt{2\pi}} \int P(x)\frac{(ix)^n}{n!}\,dx = \frac{(i)^n M_n}{n!\sqrt{2\pi}}. \tag{A.14}$$

M_n is the nth moment of $P(x)$. Consider the probability distribution of $z = x_1 + x_2$,

$$P_2(z) = \int P(x_1)P(z - x_1)dx_1. \tag{A.15}$$

By the Fourier convolution theorem, the characteristic function of $P_2(z)$ is simply $C_2(k) = \sqrt{2\pi}C(k)^2$. Consider in general the probability distribution of $z = x_1 + x_2 + \cdots + x_n$.

$$P_n(z) = \int P(x_n)P_{n-1}(z - x_n)dx_n. \tag{A.16}$$

Again by the Fourier convolution theorem, the characteristic function of $P_n(z)$ is simply $C_n(k) = \sqrt{2\pi}C(k)C_{n-1}(k)$. Clearly, by mathematical induction, $C_n(k) = (\sqrt{2\pi})^{n-1}C(k)^n$. Therefore the probability distribution of $y = z/n$ is $nP_n(ny)$ and the corresponding characteristic function is

$$C_n\left(\frac{k}{n}\right) = \frac{1}{\sqrt{2\pi}}\left[1 - \frac{\sigma_x^2 k^2}{2n^2} + o\left(\frac{1}{n^2}\right)\right]^n. \tag{A.17}$$

Since

$$\left(1 + \frac{c}{n^2}\right)^n = \exp\left[n \ln\left(1 + \frac{c}{n^2}\right)\right] = \exp\left[\frac{c}{n} + o\left(\frac{1}{n}\right)\right], \tag{A.18}$$

$$C_n\left(\frac{k}{n}\right) = \frac{1}{\sqrt{2\pi}}\exp\left[\frac{-\sigma_x^2 k^2}{2n} + o\left(\frac{1}{n}\right)\right] \to \frac{1}{\sqrt{2\pi}}\exp\left(\frac{-\sigma_x^2 k^2}{2n}\right). \tag{A.19}$$

By inverse Fourier transformation,

$$nP_n(ny) \to \sqrt{\frac{n}{2\pi}}\frac{1}{\sigma_x}\exp\left(\frac{-ny^2}{2\sigma_x^2}\right). \tag{A.20}$$

Eq. (A.20) shows the probability distribution of the average measurement result is a Gaussian distribution with a variance σ_x^2/n. Note that we did not assume x has a Gaussian distribution. The central limit theorem tells us that for any reasonable distribution, the average result is a Gaussian distribution with a variance n-fold smaller than that of $P(x)$.

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