

## CHAPTER 6

---

# FINITE-DIFFERENCE TIME-DOMAIN METHOD

---

In the preceding chapters, steady-state wave equations were solved in which the derivative with respect to the time  $t$  (i.e.,  $\partial/\partial t$ ) was replaced by  $j\omega$ . In this chapter, we discuss the finite-difference time-domain method (FD-TDM), which was developed by Yee [1] and which directly solves time-dependent Maxwell equations. The FD-TDM was originally proposed for electromagnetic waves with long wavelengths, such as microwaves, because the spatial discretization it requires is small ( $\frac{1}{10}$ – $\frac{1}{20}$  of the wavelength). As the FD-TDM is an explicit scheme, the time step in the calculation is defined by the spatial discretization width. Thus, the time step in the optical waveguide analysis is extremely short when wavelengths are of micrometer order. The amount of required memory is enormous for 3D structures, but the method is readily applicable to 2D structures. Finite-difference TDM CAD software suitable for microwave wavelengths as well as optical wavelengths is available on the market.

### 6.1 DISCRETIZATION OF ELECTROMAGNETIC WAVES

The 3D formulation is shown here because it is more versatile than the 2D formulation, which can be easily obtained from the 3D formulation.

The time-dependent Maxwell equations are

$$-\mu_0 \frac{\partial \mathbf{H}}{\partial t} = \nabla \times \mathbf{E}, \quad (6.1)$$

$$\varepsilon_0 \varepsilon_r \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{H}, \quad (6.2)$$

and using Eqs. (2.5)–(2.10), we can write their component representations as follows:

$$-\mu_0 \frac{\partial H_x}{\partial t} = \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}, \quad (6.3)$$

$$-\mu_0 \frac{\partial H_y}{\partial t} = \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}, \quad (6.4)$$

$$-\mu_0 \frac{\partial H_z}{\partial t} = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}, \quad (6.5)$$

$$\varepsilon_0 \varepsilon_r \frac{\partial E_x}{\partial t} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}, \quad (6.6)$$

$$\varepsilon_0 \varepsilon_r \frac{\partial E_y}{\partial t} = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}, \quad (6.7)$$

$$\varepsilon_0 \varepsilon_r \frac{\partial E_z}{\partial t} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}. \quad (6.8)$$

When we assume that  $\Delta x$ ,  $\Delta y$ , and  $\Delta z$  are spatial discretizations and that  $\Delta t$  is a time step, the function  $F(x, y, z, t)$  is discretized as

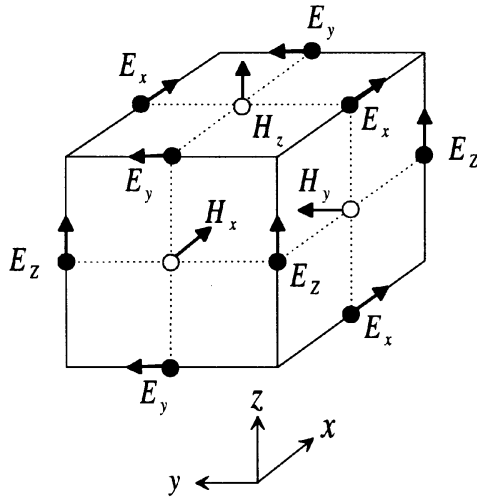
$$F^n(i, j, k) = F(i\Delta x, j\Delta y, k\Delta z, n\Delta t) = F(x, y, z, t). \quad (6.9)$$

Figure 6.1 shows what we call the Yee lattice [1]. Using  $\alpha$  to represent a spatial coordinate such as  $x$ ,  $y$ , and  $z$ , we define

$$E_\alpha = \begin{cases} \text{spatial coordinate } \alpha: \text{ half-integer,} \\ \text{the other spatial ones: integer,} \\ \text{time: integer,} \end{cases} \quad (6.10)$$

$$H_\alpha = \begin{cases} \text{spatial coordinate } \alpha: \text{ integer,} \\ \text{the other spatial ones: half-integer,} \\ \text{time: half-integer,} \end{cases} \quad (6.11)$$

Difference centers play important roles when Eqs. (6.3)–(6.8) are discretized, and here we investigate Eq. (6.3) as an example. Since the spatial difference centers of the left-hand side of the equation are the same



**FIGURE 6.1.** Yee lattice.

as those for  $H_x$ , the spatial difference centers in the  $x$ ,  $y$ , and  $z$  directions are respectively found to be  $x = i \Delta x$ ,  $y = (j + \frac{1}{2}) \Delta y$ , and  $z = (k + \frac{1}{2}) \Delta z$ . Since the time difference center of the right-hand side of the equation is the same as that for the electric fields  $E_y$  and  $E_z$ , we can write  $t = n \Delta t$ . Here,  $i, j, k$ , and  $n$  are integers. In a similar manner, we can also obtain the following difference centers of the spatial coordinates and the time for Eqs. (6.3)–(6.8).

$$\begin{aligned} \text{Eq. (6.3): } \quad x &= i \Delta x, & y &= (j + \frac{1}{2}) \Delta y, & z &= (k + \frac{1}{2}) \Delta z, \\ & t &= n \Delta t, \end{aligned} \tag{6.12}$$

$$\begin{aligned} \text{Eq. (6.4): } \quad x &= (i + \frac{1}{2}) \Delta x, & y &= j \Delta y, & z &= (k + \frac{1}{2}) \Delta z, \\ & t &= n \Delta t, \end{aligned} \tag{6.13}$$

$$\begin{aligned} \text{Eq. (6.5): } \quad x &= (i + \frac{1}{2}) \Delta x, & y &= (j + \frac{1}{2}) \Delta y, & z &= k \Delta z, \\ & t &= n \Delta t, \end{aligned} \tag{6.14}$$

$$\begin{aligned} \text{Eq. (6.6): } \quad x &= (i + \frac{1}{2}) \Delta x, & y &= j \Delta y, & z &= k \Delta z, \\ & t &= (n + \frac{1}{2}) \Delta t, \end{aligned} \tag{6.15}$$

$$\begin{aligned} \text{Eq. (6.7): } \quad x &= i \Delta x, & y &= (j + \frac{1}{2}) \Delta y, & z &= k \Delta z, \\ & t &= (n + \frac{1}{2}) \Delta t, \end{aligned} \tag{6.16}$$

$$\begin{aligned} \text{Eq. (6.8): } \quad x &= i \Delta x, & y &= j \Delta y, & z &= (k + \frac{1}{2}) \Delta z, \\ & t &= (n + \frac{1}{2}) \Delta t. \end{aligned} \tag{6.17}$$

The 3D finite-difference time-domain expressions for Eqs. (6.3)–(6.8) can be obtained by discretizing them on the basis of the difference centers (6.12)–(6.17). Again, we investigate Eq. (6.3) as an example. The difference centers  $x$  and  $t$  are both integers and the difference centers  $y$  and  $z$  are both half-integers. Thus, for the left-hand side of Eq. (6.3), we get

$$-\frac{\mu_0}{\Delta t} [H_x^{n+1/2}(i, j + \frac{1}{2}, k + \frac{1}{2}) - H_x^{n-1/2}(i, j + \frac{1}{2}, k + \frac{1}{2})]. \quad (6.18)$$

For the right-hand side, we get

$$\begin{aligned} & \frac{1}{\Delta y} [E_z^n(i, j + 1, k + \frac{1}{2}) - E_z^n(i, j, k + \frac{1}{2})] \\ & - \frac{1}{\Delta z} [E_y^n(i, j + \frac{1}{2}, k + 1) - E_y^n(i, j + \frac{1}{2}, k)]. \end{aligned} \quad (6.19)$$

Using expressions (6.18) and (6.19), we get the following finite-difference time-domain expression for Eq. (6.3):

$$\begin{aligned} H_x^{n+1/2}(i, j + \frac{1}{2}, k + \frac{1}{2}) &= H_x^{n-1/2}(i, j + \frac{1}{2}, k + \frac{1}{2}) \\ & - \frac{\Delta t}{\mu_0} \left\{ \frac{1}{\Delta y} [E_z^n(i, j + 1, k + \frac{1}{2}) - E_z^n(i, j, k + \frac{1}{2})] \right. \\ & \left. - \frac{1}{\Delta z} [E_y^n(i, j + \frac{1}{2}, k + 1) - E_y^n(i, j + \frac{1}{2}, k)] \right\}. \end{aligned} \quad (6.20)$$

Through the same procedure, we get the following finite-difference time-domain expressions for the  $y$  and  $z$  components of the magnetic fields:

$$\begin{aligned} H_y^{n+1/2}(i + \frac{1}{2}, j, k + \frac{1}{2}) &= H_y^{n-1/2}(i + \frac{1}{2}, j, k + \frac{1}{2}) \\ & - \frac{\Delta t}{\mu_0} \left\{ \frac{1}{\Delta z} [E_x^n(i + \frac{1}{2}, j, k + 1) - E_x^n(i + \frac{1}{2}, j, k)] \right. \\ & \left. - \frac{1}{\Delta x} [E_z^n(i + 1, j, k + \frac{1}{2}) - E_z^n(i, j, k + \frac{1}{2})] \right\}. \end{aligned} \quad (6.21)$$

and

$$\begin{aligned}
 H_z^{n+1/2}(i + \frac{1}{2}, j + \frac{1}{2}, k) &= H_z^{n-1/2}(i + \frac{1}{2}, j + \frac{1}{2}, k) \\
 &\quad - \frac{\Delta t}{\mu_0} \left\{ \frac{1}{\Delta x} [E_y^n(i + 1, j + \frac{1}{2}, k) - E_y^n(i, j + \frac{1}{2}, k)] \right. \\
 &\quad \left. - \frac{1}{\Delta y} [E_x^n(i + \frac{1}{2}, j + 1, k) - E_x^n(i + \frac{1}{2}, j, k)] \right\}.
 \end{aligned} \tag{6.22}$$

Next, we discretize Eq. (6.6). According to expression (6.15), the difference centers of  $x$  and  $t$  are both half-integers and the difference centers  $y$  and  $z$  are both integers. Thus, for the left-hand side of Eq. (6.6), we get

$$\frac{\varepsilon_0 \varepsilon_r}{\Delta t} [E_x^{n+1}(i + \frac{1}{2}, j, k) - E_x^n(i + \frac{1}{2}, j, k)]. \tag{6.23}$$

For the right-hand side, we get

$$\begin{aligned}
 &\frac{1}{\Delta y} [H_z^{n+1/2}(i + \frac{1}{2}, j + \frac{1}{2}, k) - H_z^{n+1/2}(i + \frac{1}{2}, j - \frac{1}{2}, k)] \\
 &\quad - \frac{1}{\Delta z} [H_y^{n+1/2}(i + \frac{1}{2}, j, k + \frac{1}{2}) - H_y^{n+1/2}(i + \frac{1}{2}, j, k - \frac{1}{2})].
 \end{aligned} \tag{6.24}$$

Using expressions (6.23) and (6.24), we get the following finite-difference time-domain expression for Eq. (6.6):

$$\begin{aligned}
 E_x^{n+1}(i + \frac{1}{2}, j, k) &= E_x^n(i + \frac{1}{2}, j, k) \\
 &\quad + \frac{\Delta t}{\varepsilon_0 \varepsilon_r} \left\{ \frac{1}{\Delta y} [H_z^{n+1/2}(i + \frac{1}{2}, j + \frac{1}{2}, k) \right. \\
 &\quad - H_z^{n+1/2}(i + \frac{1}{2}, j - \frac{1}{2}, k)] \\
 &\quad - \frac{1}{\Delta z} [H_y^{n+1/2}(i + \frac{1}{2}, j, k + \frac{1}{2}) \\
 &\quad \left. - H_y^{n+1/2}(i + \frac{1}{2}, j, k - \frac{1}{2})] \right\}.
 \end{aligned} \tag{6.25}$$

Through the same procedure, we get the following finite-difference time-domain expressions for the  $y$  and  $z$  components of the electric fields:

$$\begin{aligned}
 E_y^{n+1}(i, j + \frac{1}{2}, k) &= E_y^n(i, j + \frac{1}{2}, k) \\
 &+ \frac{\Delta t}{\varepsilon_0 \varepsilon_r} \left\{ \frac{1}{\Delta z} [H_x^{n+1/2}(i, j + \frac{1}{2}, k + \frac{1}{2}) \right. \\
 &- H_x^{n+1/2}(i, j + \frac{1}{2}, k - \frac{1}{2})] \\
 &- \frac{1}{\Delta x} [H_z^{n+1/2}(i + \frac{1}{2}, j + \frac{1}{2}, k) \\
 &- H_z^{n+1/2}(i - \frac{1}{2}, j + \frac{1}{2}, k)] \left. \right\} \quad (6.26)
 \end{aligned}$$

and

$$\begin{aligned}
 E_z^{n+1}(i, j, k + \frac{1}{2}) &= E_z^n(i, j, k + \frac{1}{2}) \\
 &+ \frac{\Delta t}{\varepsilon_0 \varepsilon_r} \left\{ \frac{1}{\Delta x} [H_y^{n+1/2}(i + \frac{1}{2}, j, k + \frac{1}{2}) \right. \\
 &- H_y^{n+1/2}(i - \frac{1}{2}, j, k + \frac{1}{2})] \\
 &- \frac{1}{\Delta y} [H_x^{n+1/2}(i, j + \frac{1}{2}, k + \frac{1}{2}) \\
 &- H_x^{n+1/2}(i, j - \frac{1}{2}, k + \frac{1}{2})] \left. \right\}. \quad (6.27)
 \end{aligned}$$

Magnetic fields  $H_x^{n+1/2}$  with the half-integer time step  $(n + 1/2) \Delta t$  are calculated first from Eqs. (6.20)–(6.22) by using the electric fields with the integer time step  $n \Delta t$ . Then those fields are used to calculate the electric fields  $E_x^{n+1}$  with the integer time step  $(n + 1) \Delta t$  by using Eqs. (6.25)–(6.27). Repeating these two steps, we can calculate the time evolution of the electric and magnetic fields directly.

It should be noted that the relative permittivity at the interface between two media is approximated better by using  $(\varepsilon_{r1} + \varepsilon_{r2})/2$  than by using only  $\varepsilon_{r1}$  or  $\varepsilon_{r2}$ .

## 6.2 STABILITY CONDITION

In an explicit scheme such as the FD-TDM, the time step  $\Delta t$  in the calculation is restricted by the spatial discretization. For simplicity in discussing the stability condition here, we will use the 1D scalar Helmholtz equation

$$\frac{\partial^2 \phi}{\partial x^2} - \varepsilon \mu_0 \frac{\partial^2 \phi}{\partial t^2} = 0, \quad (6.28)$$

where  $\phi$  is a 1D wave function that designates the time-dependent field. Using  $\beta_x$  as the  $x$ -directed propagation constant, we express this wave function as

$$\begin{aligned} \phi(x, t) &= \exp(j\beta_x x) \exp(\alpha t) \\ &= \exp(j\beta_x p \Delta x) \exp(\alpha n \Delta t) \\ &= \exp(j\beta_x p \Delta x) \xi^n, \end{aligned} \quad (6.29)$$

where  $\xi = \exp(\alpha \Delta t)$ . Thus, if the field is to be stable,  $\xi$  has to satisfy the condition

$$|\xi| \leq 1. \quad (6.30)$$

Substituting Eq. (6.29) into (6.28), we get

$$\begin{aligned} &\frac{1}{(\Delta x)^2} \{ \exp[j\beta_x(p+1)\Delta x] \xi^n - 2 \exp(j\beta_x p \Delta x) \xi^n + \exp[j\beta_x(p-1)\Delta x] \xi^n \} \\ &- \frac{\varepsilon \mu_0}{(\Delta t)^2} \{ \exp(j\beta_x p \Delta x) \xi^{n+1} - 2 \exp(j\beta_x p \Delta x) \xi^n + \exp(j\beta_x p \Delta x) \xi^{n-1} \} = 0. \end{aligned}$$

Dividing this equation by  $\exp(j\beta_x p \Delta x) \xi^n$ , we reduce it to

$$\frac{1}{(\Delta x)^2} \{ \exp(j\beta_x \Delta x) - 2 + \exp(-j\beta_x \Delta x) \} - \frac{\varepsilon \mu_0}{(\Delta t)^2} (\xi - 2 + \xi^{-1}) = 0. \quad (6.31)$$

Dividing Eq. (6.31) by  $\varepsilon\mu_0/(\Delta t)^2\xi$  and considering that the first term of Eq. (6.31) can be rewritten as

$$\begin{aligned}\{\exp(j\beta_x \Delta x) - 2 + \exp(-j\beta_x \Delta x)\} &= 2(\cos(\beta_x \Delta x) - 1) \\ &= -4 \sin^2\left(\beta_x \frac{\Delta x}{2}\right),\end{aligned}$$

we get

$$(\xi^2 - 2\xi + 1) - \frac{(\Delta t)^2}{\varepsilon\mu_0} \left[ -\frac{4}{(\Delta x)^2} \sin^2\left(\beta_x \frac{\Delta x}{2}\right) \right] \xi = 0$$

and therefore

$$\xi^2 - 2A + 1 = 0, \quad (6.32)$$

where the parameter  $A$  is defined by

$$A = -\frac{2(\Delta t)^2}{\varepsilon\mu_0} \frac{1}{(\Delta x)^2} \sin^2\left(\beta_x \frac{\Delta x}{2}\right) + 1. \quad (6.33)$$

The roots of Eq. (6.32) are

$$\xi_1 = A + \sqrt{A^2 - 1}, \quad (6.34)$$

$$\xi_2 = A - \sqrt{A^2 - 1}. \quad (6.35)$$

Because  $|\xi| \leq 1$  and  $0 \leq \sin^2 \theta$ , we get the relation

$$A = -\frac{2(\Delta t)^2}{\varepsilon\mu_0} \frac{1}{(\Delta x)^2} \sin^2\left(\beta_x \frac{\Delta x}{2}\right) + 1 \leq 1. \quad (6.36)$$

We can thus specify the stability condition in terms of  $A$ :

*Case  $A < -1$ .* Since, according to Eq. (6.35),  $1 < |\xi_2|$ , the field is unstable.

Case  $-1 \leq A \leq 1$ . Since  $\xi_1$  and  $\xi_2$  can be expressed as

$$\xi_1 = A + \sqrt{A^2 - 1} = A + j\sqrt{1 - A^2}, \quad (6.37)$$

$$\xi_2 = A - \sqrt{A^2 - 1} = A - j\sqrt{1 - A^2}, \quad (6.38)$$

their absolute values can be expressed as

$$|\xi_1| = |\xi_2| = A^2 + (1 - A^2) = 1. \quad (6.39)$$

Thus, the field is stable when

$$-1 \leq A \leq 1. \quad (6.40)$$

Relation (6.40) can be interpreted as imposing the following restriction on the time step (see Problem 1):

$$\Delta t \leq \sqrt{\varepsilon\mu_0} \left( \frac{1}{(\Delta x)^2} \right)^{-1/2} = \frac{\sqrt{\varepsilon_r}}{c_0} \left( \frac{1}{(\Delta x)^2} \right)^{-1/2} = \frac{1}{v} \left( \frac{1}{(\Delta x)^2} \right)^{-1/2}, \quad (6.41)$$

where  $\Delta x$  is the spatial discretization width and  $\Delta t$  is the time step and where  $c_0$ ,  $\varepsilon_r$ , and  $v = c_0/\sqrt{\varepsilon_r}$  are respectively the velocity of the light in a vacuum, the relative permittivity of the medium, and the velocity of the light in the medium. Equation (6.41) is for a 1D structure, and the corresponding restriction for a 3D structure is

$$\Delta t \leq \frac{1}{v} \left( \frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2} + \frac{1}{(\Delta z)^2} \right)^{-1/2}. \quad (6.42)$$

### 6.3 ABSORBING BOUNDARY CONDITIONS

Since the FD-TDM, like the BPM in Chapter 5, has finite analysis windows, an artificial boundary condition suppressing reflections at the analysis windows is required. Mur's absorbing boundary condition (ABC) [2] is often used for this purpose, though Berenger's perfectly matched layer (PML) scheme [3] has also come into use recently. The PML scheme suppresses reflections better than Mur's ABC does, but Mur's condition is easier to use. Here, we discuss Mur's first-order ABC.

As shown in Fig. 6.2, the analysis window is defined in ranges of  $(0, L_x)$  in the  $x$  direction,  $(0, L_y)$  in the  $y$  direction, and  $(0, L_z)$  in the  $z$  direction.

1.  $x = 0$ :  $E_y$  and  $E_z$ . The electric fields  $E_y$  and  $E_z$  are on the boundary  $x = 0$ . The wave function  $W$  for the left-traveling wave incident perpendicular to the boundary is

$$W = \exp[j(\omega t + \beta_x x)] = \exp\left[j\omega\left(t + \frac{1}{v_x}x\right)\right], \quad (6.43)$$

where  $v_x$  is the velocity of the wave. Thus, the derivatives of the wave function with respect to  $x$  and  $t$  are

$$\frac{\partial W}{\partial x} = j\omega \frac{1}{v_x} \exp\left[j\omega\left(t + \frac{1}{v_x}x\right)\right] = \frac{1}{v_x} j\omega W, \quad (6.44)$$

$$\frac{\partial W}{\partial t} = j\omega W. \quad (6.45)$$

Substituting Eq. (6.45) into (6.44), we get

$$\frac{\partial W}{\partial x} = \frac{1}{v_x} \frac{\partial W}{\partial t}$$

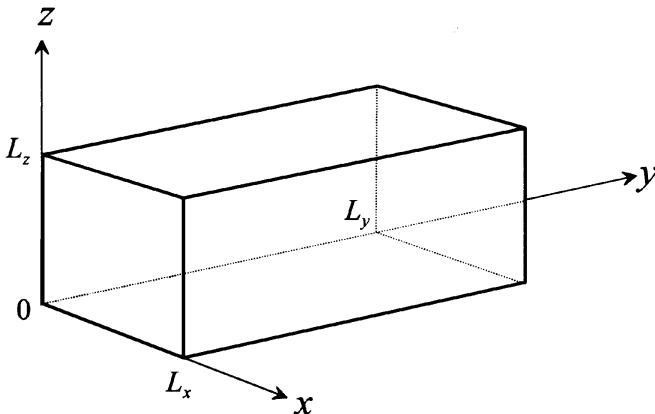


FIGURE 6.2. Analysis region.

and therefore

$$\frac{\partial W}{\partial x} - \frac{1}{v_x} \frac{\partial W}{\partial t} = 0.$$

Thus, the wave equation for Mur's first-order ABC is

$$\left( \frac{\partial}{\partial x} - \frac{1}{v_x} \frac{\partial}{\partial t} \right) W \Big|_{x=0} = 0. \quad (6.46)$$

Next, we discretize the wave equation (6.46) for the case that the wave function is that for the  $y$ -directed electric field  $E_y$ . Assuming that the node number of the node on the boundary is 0, we discretize the derivative of the electric field  $E_y$  with respect to the coordinate  $x$  as

$$\begin{aligned} \frac{\partial E_y}{\partial x} = \frac{1}{2} \left\{ \frac{1}{\Delta x} [E_y^n(1, j + \frac{1}{2}, k) - E_y^n(0, j + \frac{1}{2}, k)] \right. \\ \left. + \frac{1}{\Delta x} [E_y^{n+1}(1, j + \frac{1}{2}, k) - E_y^{n+1}(0, j + \frac{1}{2}, k)] \right\}, \quad (6.47) \end{aligned}$$

where the time average between  $n$  and  $n + 1$  was taken on the right-hand. On the other hand, we discretize the derivative of the electric field  $E_y$  with respect to time  $t$  as

$$\begin{aligned} \frac{\partial E_y}{\partial t} = \frac{1}{2} \left\{ \frac{1}{\Delta t} [E_y^{n+1}(1, j + \frac{1}{2}, k) - E_y^n(1, j + \frac{1}{2}, k)] \right. \\ \left. + \frac{1}{\Delta t} [E_y^{n+1}(0, j + \frac{1}{2}, k) - E_y^n(0, j + \frac{1}{2}, k)] \right\}, \quad (6.48) \end{aligned}$$

where the spatial average between  $i = 0$  and  $i = 1$  was taken. Substituting Eqs. (6.47) and (6.48) into Eq. (6.46), we can derive the following finite-difference time-domain expression for the electric field  $E_y$ :

$$\begin{aligned} E_y^{n+1}(0, j + \frac{1}{2}, k) = E_y^n(1, j + \frac{1}{2}, k) \\ + \frac{v_x \Delta t - \Delta x}{v_x \Delta t + \Delta x} [E_y^{n+1}(1, j + \frac{1}{2}, k) - E_y^n(0, j + \frac{1}{2}, k)]. \quad (6.49) \end{aligned}$$

We can similarly derive the finite-difference time-domain expression for the electric field  $E_z$ :

$$E_z^{n+1}(0, j, k + \frac{1}{2}) = E_z^n(1, j, k + \frac{1}{2}) + \frac{v_x \Delta t - \Delta x}{v_x \Delta t + \Delta x} [E_z^{n+1}(1, j, k + \frac{1}{2}) - E_z^n(0, j, k + \frac{1}{2})]. \quad (6.50)$$

2.  $x = L_x$ :  $E_y$  and  $E_z$ . The electric fields  $E_y$  and  $E_z$  are on the boundary  $x = L_x$ . The wave function  $W$  for the right-traveling wave incident perpendicular to the boundary is

$$W = \exp[j(\omega t - \beta_x x)] = \exp\left[j\omega\left(t - \frac{1}{v_x}x\right)\right], \quad (6.51)$$

where  $v_x$  is a velocity of the wave. Thus, the derivatives of the wave function with respect to  $x$  and  $t$  are

$$\frac{\partial W}{\partial x} = -\frac{1}{v_x} j\omega W, \quad (6.52)$$

$$\frac{\partial W}{\partial t} = j\omega W. \quad (6.53)$$

Substituting Eq. (6.53) into (6.52), we get the wave equation

$$\left(\frac{\partial}{\partial x} + \frac{1}{v_x} \frac{\partial}{\partial t}\right) W \Big|_{x=L_x} = 0. \quad (6.54)$$

Next, we discretize the wave equation (6.46) for the case that the wave function is the  $y$ -directed electric field  $E_y$ . Assuming that the node number of the node on the boundary is  $N_x$ , we discretize the derivative of the electric field  $E_y$  with respect to  $x$  as

$$\frac{\partial E_y}{\partial x} = \frac{1}{2} \left\{ \frac{1}{\Delta x} [E_y^n(N_x, j + \frac{1}{2}, k) - E_y^n(N_x - 1, j + \frac{1}{2}, k)] + \frac{1}{\Delta x} [E_y^{n+1}(N_x, j + \frac{1}{2}, k) - E_y^{n+1}(N_x - 1, j + \frac{1}{2}, k)] \right\}, \quad (6.55)$$

where the time average between  $n$  and  $n + 1$  was taken on the right-hand side. On the other hand, we discretize the derivative of the electric field  $E_y$  with respect to time  $t$  as

$$\begin{aligned} \frac{\partial E_y}{\partial t} = \frac{1}{2} \left\{ \frac{1}{\Delta t} [E_y^{n+1}(N_x, j + \frac{1}{2}, k) - E_y^n(N_x, j + \frac{1}{2}, k)] \right. \\ \left. + \frac{1}{\Delta t} [E_y^{n+1}(N_x - 1, j + \frac{1}{2}, k) - E_y^n(N_x - 1, j + \frac{1}{2}, k)] \right\}, \quad (6.56) \end{aligned}$$

where the spatial average between  $i = N_x$  and  $N_x - 1$  was taken. Substituting Eqs. (6.55) and (6.56) into Eq. (6.54), we can derive the finite-difference time-domain expression for the electric field  $E_y$ :

$$\begin{aligned} E_y^{n+1}(N_x, j + \frac{1}{2}, k) = E_y^n(N_x - 1, j + \frac{1}{2}, k) \\ + \frac{v_x \Delta t - \Delta x}{v_x \Delta t + \Delta x} [E_y^{n+1}(N_x - 1, j + \frac{1}{2}, k) \\ - E_y^n(N_x, j + \frac{1}{2}, k)]. \quad (6.57) \end{aligned}$$

We can similarly derive the finite-difference time-domain expression for the electric field  $E_z$ :

$$\begin{aligned} E_z^{n+1}(N_x, j, k + \frac{1}{2}) = E_z^n(N_x - 1, j, k + \frac{1}{2}) \\ + \frac{v_x \Delta t - \Delta x}{v_x \Delta t + \Delta x} [E_z^{n+1}(N_x - 1, j, k + \frac{1}{2}) \\ - E_z^n(N_x, j, k + \frac{1}{2})]. \quad (6.58) \end{aligned}$$

For the ABCs on  $y = 0$  and  $y = L_y$  and on  $z = 0$  and  $z = L_z$ , see Problem 2.

## PROBLEMS

1. Derive the restriction on the time step  $\Delta t$  specified in Eq. (6.41) by using the stability condition  $-1 \leq A \leq 1$  in relation (6.40).

**ANSWER**

Using Eq. (6.33), we can rewrite the stability condition in relation (6.40) as

$$-1 \leq -\frac{2(\Delta t)^2}{\varepsilon\mu_0} \frac{1}{(\Delta x)^2} \sin^2\left(\beta_x \frac{\Delta x}{2}\right) + 1 \leq 1. \quad (\text{P6.1})$$

Since the relation between the center term and right-hand term is always satisfied, we have to consider only the left-hand term and center term:

$$-1 \leq -\frac{2(\Delta t)^2}{\varepsilon\mu_0} \frac{1}{(\Delta x)^2} \sin^2\left(\beta_x \frac{\Delta x}{2}\right) + 1.$$

Multiplying both sides by  $-1$  and considering the case in which the right-hand side reaches its maximum, when  $\sin^2(\cdot) = 1$ , we can rewrite the above relation as

$$1 \geq \frac{2(\Delta t)^2}{\varepsilon\mu_0} \frac{1}{(\Delta x)^2} - 1.$$

Thus, we get

$$2 \geq \frac{2(\Delta t)^2}{\varepsilon\mu_0} \frac{1}{(\Delta x)^2},$$

and therefore

$$(\Delta t)^2 \leq \varepsilon\mu_0 \left(\frac{1}{(\Delta x)^2}\right)^{-1}.$$

And this relation can be rewritten as the restriction on the time step  $\Delta t$ :

$$\Delta t \leq \sqrt{\varepsilon\mu_0} \left(\frac{1}{(\Delta x)^2}\right)^{-1/2} = \frac{\sqrt{\varepsilon_r}}{c_0} \left(\frac{1}{(\Delta x)^2}\right)^{-1/2} = \frac{1}{v} \left(\frac{1}{(\Delta x)^2}\right)^{-1/2}. \quad (\text{P6.2})$$

2. Derive the ABC fields for  $y = 0$  and  $y = L_y$ , and  $z = 0$  and  $z = L_z$ .

**ANSWER**

**a.  $y = 0$ .** The wave equation on this boundary is

$$\left( \frac{\partial}{\partial y} - \frac{1}{v_y} \frac{\partial}{\partial t} \right) W \Big|_{y=0} = 0. \quad (\text{P6.3})$$

The finite-difference time-domain expressions for the electric fields  $E_x$  and  $E_z$  on this boundary are

$$\begin{aligned} E_x^{n+1}(i + \frac{1}{2}, 0, k) &= E_x^n(i + \frac{1}{2}, 1, k) \\ &+ \frac{v_y \Delta t - \Delta y}{v_y \Delta t + \Delta y} [E_x^{n+1}(i + \frac{1}{2}, 1, k) - E_x^n(i + \frac{1}{2}, 0, k)], \end{aligned} \quad (\text{P6.4})$$

$$\begin{aligned} E_z^{n+1}(i, 0, k + \frac{1}{2}) &= E_z^n(i, 1, k + \frac{1}{2}) \\ &+ \frac{v_y \Delta t - \Delta y}{v_y \Delta t + \Delta y} [E_z^{n+1}(i, 1, k + \frac{1}{2}) - E_z^n(i, 0, k + \frac{1}{2})]. \end{aligned} \quad (\text{P6.5})$$

**b.  $y = L_y$ .** The wave equation on this boundary is

$$\left( \frac{\partial}{\partial y} + \frac{1}{v_y} \frac{\partial}{\partial t} \right) W \Big|_{y=L_y} = 0. \quad (\text{P6.6})$$

The finite-difference time-domain expressions for the electric fields  $E_x$  and  $E_z$  on this boundary are

$$\begin{aligned} E_x^{n+1}(i + \frac{1}{2}, N_y, k) &= E_x^n(i + \frac{1}{2}, N_y - 1, k) \\ &+ \frac{v_y \Delta t - \Delta y}{v_y \Delta t + \Delta y} [E_x^{n+1}(i + \frac{1}{2}, N_y - 1, k) - E_x^n(i + \frac{1}{2}, N_y, k)], \end{aligned} \quad (\text{P6.7})$$

$$\begin{aligned} E_z^{n+1}(i, N_y, k + \frac{1}{2}) &= E_z^n(i, N_y - 1, k + \frac{1}{2}) \\ &+ \frac{v_y \Delta t - \Delta y}{v_y \Delta t + \Delta y} [E_z^{n+1}(i, N_y - 1, k + \frac{1}{2}) - E_z^n(i, N_y, k + \frac{1}{2})]. \end{aligned} \quad (\text{P6.8})$$

c.  $z = 0$ . The wave equation on this boundary is

$$\left( \frac{\partial}{\partial z} - \frac{1}{v_z} \frac{\partial}{\partial t} \right) W \Big|_{z=0} = 0.$$

The finite-difference time-domain expressions for the electric fields  $E_x$  and  $E_y$  on this boundary are

$$\begin{aligned} E_x^{n+1}(i + \frac{1}{2}, j, 0) &= E_x^n(i + \frac{1}{2}, j, 1) \\ &+ \frac{v_z \Delta t - \Delta z}{v_z \Delta t + \Delta z} [E_x^{n+1}(i + \frac{1}{2}, j, 1) - E_x^n(i + \frac{1}{2}, j, 0)], \end{aligned} \quad (\text{P6.9})$$

$$\begin{aligned} E_y^{n+1}(i, j + \frac{1}{2}, 0) &= E_y^n(i, j + \frac{1}{2}, 1) \\ &+ \frac{v_z \Delta t - \Delta y}{v_z \Delta t + \Delta y} [E_y^{n+1}(i, j + \frac{1}{2}, 1) - E_y^n(i, j + \frac{1}{2}, 0)]. \end{aligned} \quad (\text{P6.10})$$

d.  $z = L_z$ . The wave equation on this boundary is

$$\left( \frac{\partial}{\partial z} + \frac{1}{v_z} \frac{\partial}{\partial t} \right) W \Big|_{z=L_z} = 0. \quad (\text{P6.11})$$

The finite-difference time-domain expressions for the electric fields  $E_x$  and  $E_y$  on this boundary are

$$\begin{aligned} E_x^{n+1}(i + \frac{1}{2}, j, N_z) &= E_x^n(i + \frac{1}{2}, j, N_z - 1) \\ &+ \frac{v_z \Delta t - \Delta z}{v_z \Delta t + \Delta z} [E_x^{n+1}(i + \frac{1}{2}, j, N_z - 1) - E_x^n(i + \frac{1}{2}, j, N_z)], \end{aligned} \quad (\text{P6.12})$$

$$\begin{aligned} E_y^{n+1}(i, j + \frac{1}{2}, N_z) &= E_y^n(i, j + \frac{1}{2}, N_z - 1) \\ &+ \frac{v_z \Delta t - \Delta y}{v_z \Delta t + \Delta y} [E_y^{n+1}(i, j + \frac{1}{2}, N_z - 1) - E_y^n(i, j + \frac{1}{2}, N_z)]. \end{aligned} \quad (\text{P6.13})$$

**REFERENCES**

- [1] K. S. Yee, "Numerical solution of initial boundary value problems involving Maxwell's equations in isotropic media," *IEEE Trans. Antennas Propagat.*, vol. AP-14, pp. 302–307, 1966.
- [2] G. Mur, "Absorbing boundary conditions for the finite-difference time-domain approximation of the time domain electromagnetic field equations," *IEEE Trans. Electromagn. Compat.*, vol. EMC-23, pp. 377–382, 1981.
- [3] J.-P. Berenger, "A perfectly matched layer for the absorption of electromagnetic waves," *J. Computat. Phys.*, vol. 114, pp. 185–220, 1994.