Introduction to Optical Waveguide Analysis: Solving Maxwell's Equations and the Schrödinger Equation. Kenji Kawano, Tsutomu Kitoh

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APPENDIX B

INTEGRATION FORMULA FOR AREA COORDINATES

The integration formula shown in Eq. (3.184) is derived here by calculating the following integration for a triangular element e shown in Fig. 3.4:

$$I_e(i, j, k) = \int\int_e L_1^i L_2^j L_3^k \, dx \, dy, \tag{B.1}$$

where i, j, and k are integers and the spatial coordinates of nodes 1, 2, and 3 are respectively (x_1, y_1), (x_2, y_2), and (x_3, y_3).

As shown in Eq. (3.72), we have the following relation between the spatial coordinates and the area coordinates:

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix}. \tag{B.2}$$

As shown by the bottom row of Eq. (B.2),

$$L_1 + L_2 + L_3 = 1. \tag{B.3}$$

According to Eq. (B.2), x and y can be expressed using L_1 and L_2 as follows:

$$\begin{aligned} x &= x_1 L_1 + x_2 L_2 + x_3 L_3 \\ &= x_1 L_1 + x_2 L_2 + x_3 (1 - L_1 - L_2) \\ &= (x_1 - x_3) L_1 + (x_2 - x_3) L_2, \end{aligned} \tag{B.4}$$

$$\begin{aligned} y &= y_1 L_1 + y_2 L_2 + y_3 L_3 \\ &= y_1 L_1 + y_2 L_2 + y_3 (1 - L_1 - L_2) \\ &= (y_1 - y_3) L_1 + (y_2 - y_3) L_2. \end{aligned} \tag{B.5}$$

Thus, we get

$$\frac{\partial(x, y)}{\partial(L_1, L_2)} = \begin{vmatrix} \dfrac{\partial x}{\partial L_1} & \dfrac{\partial x}{\partial L_2} \\ \dfrac{\partial y}{\partial L_1} & \dfrac{\partial y}{\partial L_2} \end{vmatrix} = \begin{vmatrix} x_1 - x_3 & x_2 - x_3 \\ y_1 - y_3 & y_2 - y_3 \end{vmatrix} = 2S_e, \tag{B.6}$$

where S_e is the area of element e. Using Eq. (B.6) to transform x and y to L_1 and L_2, we can rewrite Eq. (B.1) as follows:

$$\begin{aligned} I_e(i, j, k) &= \iint_e L_1^i L_2^j L_3^k \, dx \, dy \\ &= \iint_e L_1^i L_2^j L_3^k \left| \frac{\partial(x, y)}{\partial(L_1, L_2)} \right| dx \, dy \\ &= 2S_e \iint_e L_1^i L_2^j L_3^k \, dL_1 \, dL_2 \\ &= 2S_e \int_0^1 dL_1 \int_0^{1-L_1} L_1^i L_2^j (1 - L_1 - L_2)^k \, dL_2 \\ &= 2S_e \int_0^1 L_1^i \, dL_1 \int_0^{1-L_1} L_2^j (1 - L_1 - L_2)^k \, dL_2. \end{aligned} \tag{B.7}$$

The second integral of Eq. (B.7) is calculated as

$$
\begin{aligned}
I_0(j,k) &= \int_0^{1-L_1} L_2^j (1-L_1-L_2)^k \, dL_2 \\
&= \left[\frac{1}{j+1} L_2^{j+1} (1-L_1-L_2)^k \right]_0^{1-L_1} \\
&\quad + \frac{k}{j+1} \int_0^{1-L_1} L_2^{j+1} (1-L_1-L_2)^{k-1} \, dL_2 \\
&= 0 + \frac{k}{j+1} \int_0^{1-L_1} L_2^{j+1} (1-L_1-L_2)^{k-1} \, dL_2 \\
&= \frac{k}{j+1} I_0(j+1, k-1) \\
&= \left[\frac{1}{j+2} L_2^{j+2} (1-L_1-L_2)^{k-1} \right]_0^{1-L_1} \\
&\quad + \frac{k(k-1)}{(j+1)(j+2)} \int_0^{1-L_1} L_2^{j+2} (1-L_1-L_2)^{k-2} \, dL_2 \\
&= 0 + \frac{k(k-1)}{(j+1)(j+2)} I_0(j+2, k-2) \\
&= \frac{k(k-1)\cdots 1}{(j+1)(j+2)\cdots(j+k)} I_0(j+k, 0) \\
&= \frac{k!j!}{(j+k)!} I_0(j+k, 0),
\end{aligned} \tag{B.8}
$$

where

$$
\begin{aligned}
I_0(j+k, 0) &= \int_0^{1-L_1} L_2^{j+k} \, dL_2 = \left[\frac{1}{j+k+1} L_2^{j+k+1} \right]_0^{1-L_1} \\
&= \frac{1}{j+k+1} (1-L_1)^{j+k+1}.
\end{aligned} \tag{B.9}
$$

Substituting Eq. (B.9) into Eq. (B.8), we get

$$
I_0(j,k) = \frac{j!k!}{(j+k+1)!} (1-L_1)^{j+k+1}. \tag{B.10}
$$

And substituting Eq. (B.10) into Eq. (B.7), we get

$$I_e(i,j,k) = 2S_e \frac{j!k!}{(j+k+1)!} \int_0^1 L_1^i (1-L_1)^{j+k+1}\, dL_1. \tag{B.11}$$

The remaining calculation is

$$\begin{aligned}
I_1(i,j+k+1) &= \int_0^1 L_1^i (1-L_1)^{j+k+1}\, dL_1 \\
&= \left[\frac{1}{i+1} L_1^{i+1} (1-L_1)^{j+k+1} \right]_0^1 \\
&\quad + \frac{j+k+1}{i+1} \int_0^1 L_1^{i+1} (1-L_1)^{j+k}\, dL_1 \\
&= 0 + \frac{j+k+1}{i+1} I_1(i+1, j+k) \\
&= \frac{(j+k+1)(j+k)\cdots 2\cdot 1}{(i+1)(i+2)\cdots(i+j+k+1)} I_1(i+j+k+1, 0) \\
&= \frac{(j+k+1)!i!}{(i+j+k+1)!} I_1(i+j+k+1, 0).
\end{aligned} \tag{B.12}$$

Substituting $I_1(i+j+k+1,0)$, where $I_1(i+j+k+1,0)$ can be rewritten as

$$\begin{aligned}
I_1(i+j+k+1, 0) &= \int_0^1 L_1^{i+j+k+1}\, dL_1 = \left[\frac{1}{i+j+k+2} L_1^{i+j+k+2} \right]_0^1 \\
&= \frac{1}{i+j+k+2},
\end{aligned} \tag{B.13}$$

into Eq. (B.12), we get

$$I_1(i,j+k+1) = \frac{(j+k+1)!i!}{(i+j+k+2)!}. \tag{B.14}$$

Then substituting this equation into Eq. (B.11), we finally get the formula

$$I_e(i, j, k) = 2S_e \frac{i!\,j!\,k!}{(i + j + k + 2)!}. \tag{B.15}$$